Série 12.

Ex. 2

$$\begin{cases} 2x+3y-z=1 \\ 4x+y-3z=11 \\ 3x-2y+5z=21 \end{cases}$$

On déf. $a_1=\begin{pmatrix}2\\4\\3\end{pmatrix}$, $a_2=\begin{pmatrix}3\\1\\-2\end{pmatrix}$, $a_3=\begin{pmatrix}-1\\-3\\+5\end{pmatrix}$

$b=\begin{pmatrix}1\\11\\21\end{pmatrix}$

$$\det([a_1|a_2|a_3]) = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 1 & -3 \\ 3 & -2 & +5 \end{vmatrix} = \quad \begin{matrix} L_1' = L_1 - 3L_2 \\ L_3' = L_3 + 2L_2 \end{matrix}$$

$$= \begin{vmatrix} -10 & 0 & +8 \\ 4 & 1 & -3 \\ 11 & 0 & -1 \end{vmatrix} \underset{\text{dév. p.r. à } C_2}{=} +1 \begin{vmatrix} -10 & 8 \\ 11 & -1 \end{vmatrix} = +10-88 = -78$$

$$\boxed{\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}}$$

$$\det([b|a_2|a_3]) = \begin{vmatrix} 1 & 3 & -1 \\ 11 & 1 & -3 \\ 21 & -2 & 5 \end{vmatrix} = \quad \begin{matrix} L_1' = L_1 - 3L_2 \\ L_3' = L_3 + 2L_2 \end{matrix}$$

$$= \begin{vmatrix} -32 & 0 & 8 \\ 11 & 1 & -3 \\ 43 & 0 & -1 \end{vmatrix} \underset{\text{dév. p.r. à } C_2}{=} +1 \begin{vmatrix} -32 & 8 \\ 43 & -1 \end{vmatrix} = -312$$

$$\det([a_1|b|a_3]) = \begin{vmatrix} 2 & 1 & -1 \\ 4 & 11 & -3 \\ 3 & 21 & 5 \end{vmatrix} \quad \begin{matrix} L_2' = L_2 - 11L_1 \\ L_3' = L_3 - 21L_1 \end{matrix}$$

$$= \begin{vmatrix} 2 & 1 & -1 \\ -18 & 0 & 8 \\ -39 & 0 & 26 \end{vmatrix} = -1 \cdot \begin{vmatrix} -18 & 8 \\ -39 & 26 \end{vmatrix} = 156$$

$$\det([a_1 | a_2 | b]) = \begin{vmatrix} 2 & 3 & 1 \\ 4 & 1 & 11 \\ 3 & -2 & 21 \end{vmatrix} = \quad \begin{matrix} L_1' = L_1 - 3L_2 \\ L_3' = L_3 + 2L_2 \end{matrix}$$

$$= \begin{vmatrix} -10 & 0 & -32 \\ 4 & 1 & 11 \\ 11 & 0 & 43 \end{vmatrix} = \begin{vmatrix} -10 & -32 \\ 11 & 43 \end{vmatrix} = \quad L_2' = L_2 + L_1$$

$$= \begin{vmatrix} -10 & -32 \\ 1 & 11 \end{vmatrix} = -110 + 32 = -78$$

$$x = \frac{\det([b | a_2 | a_3])}{\det(A)} = \frac{-312}{-78} = +4$$

$$y = \frac{\det([a_1 | b | a_3])}{\det(A)} = \frac{156}{-78} = -2$$

$$z = \frac{\det([a_1 | a_2 | b])}{\det(A)} = \frac{-78}{-78} = +1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} b$$

Ex. 3 Matrice de Vandermonde

$$V_n(\alpha_1,\dots,\alpha_n) = \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-2} & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-2} & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-2} & \alpha_n^{n-1} \end{vmatrix}$$

matrice $n \times n$

M.q. $V_n(\alpha_1,\dots,\alpha_n) = \prod_{1 \leq i < j \leq n} (\alpha_j - \alpha_i)$ (*)

Indice : $C_j \leftarrow C_j - \alpha_1 C_{j-1}$ pour $j = n, \dots, 2$

Preuve : par récurrence.

Base : $n = 2$ $\quad V_2(\alpha_1,\alpha_2) = \begin{vmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \end{vmatrix} = \alpha_2 - \alpha_1$

✓ ok

Pas de récurrence : On suppose (*) vraie au rang $n-1$ et on montre qu'elle reste vraie au rang n.

$C_n \leftarrow C_n - \alpha_1 C_{n-1}$

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-2} & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-2} & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-2} & \alpha_n^{n-1} \end{vmatrix} = \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-2} & 0 \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-2} & \alpha_2^{n-2}(\alpha_2-\alpha_1) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-2} & \alpha_n^{n-2}(\alpha_n-\alpha_1) \end{vmatrix}$$

$$= \dots = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \alpha_2-\alpha_1 & \alpha_2(\alpha_2-\alpha_1) & \cdots & \alpha_2^{n-3}(\alpha_2-\alpha_1) & \alpha_2^{n-2}(\alpha_2-\alpha_1) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \alpha_n-\alpha_1 & \alpha_n(\alpha_n-\alpha_1) & \cdots & \alpha_n^{n-3}(\alpha_n-\alpha_1) & \alpha_n^{n-2}(\alpha_n-\alpha_1) \end{vmatrix}$$

dév. p.r. à L_1 :

$$= \begin{vmatrix} \alpha_2-\alpha_1 & \alpha_2(\alpha_2-\alpha_1) & \cdots & \alpha_2^{n-3}(\alpha_2-\alpha_1) & \alpha_2^{n-2}(\alpha_2-\alpha_1) \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_n-\alpha_1 & \alpha_n(\alpha_n-\alpha_1) & \cdots & \alpha_n^{n-3}(\alpha_n-\alpha_1) & \alpha_n^{n-2}(\alpha_n-\alpha_1) \end{vmatrix}$$

par linéarité du dét. p.r. aux lignes

$$= (\alpha_2-\alpha_1)(\alpha_3-\alpha_1)\cdots(\alpha_n-\alpha_1)\cdot \begin{vmatrix} 1 & \alpha_2 & \cdots & \alpha_2^{n-3} & \alpha_2^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_n & \cdots & \alpha_n^{n-3} & \alpha_n^{n-2} \end{vmatrix}$$

$$= \prod_{k=2}^{n}(\alpha_k-\alpha_1)\cdot V_{n-1}(\alpha_2,\dots,\alpha_n) =$$ par le pas de récurrence $$=$$

$$= \prod_{k=2}^{n}(\alpha_k-\alpha_1)\cdot \prod_{2\leq i<j\leq n}(\alpha_j-\alpha_i) = \prod_{1\leq i<j\leq n}(\alpha_j-\alpha_i) \quad \blacksquare$$

Ex. 4

1. M.q.

$$\begin{array}{cc} & \begin{array}{cc} k_1 & k_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array} \cdot \begin{array}{cc} & \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{array}{c} k_1 \\ k_2 \end{array} & \begin{pmatrix} E & F \\ G & H \end{pmatrix} \end{array} = \begin{pmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{pmatrix}$$

<u>Solution</u>: On note $M := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ et $N := \begin{pmatrix} E & F \\ G & H \end{pmatrix}$

On montre que le premier bloc $AE+BG$, c.-à-d., $1 \leq i \leq m_1$ et $1 \leq j \leq p_1$:

$$(MN)_{i,j} = \sum_{t=1}^{k_1+k_2} M_{i,t} \cdot N_{t,j} = \sum_{t=1}^{k_1} M_{i,t} N_{t,j} + \sum_{t=k_1+1}^{k_1+k_2} M_{i,t} N_{t,j}$$

$$= \sum_{t=1}^{k_1} A_{i,t} E_{t,j} + \sum_{t=1}^{k_2} B_{i,t} \cdot G_{t,j}$$

$$= (AE)_{i,j} + (BG)_{i,j}$$

2. (a) M.q. $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix} \cdot \begin{pmatrix} I_n & 0 \\ 0 & D \end{pmatrix}$

et $\det\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \det(A) \cdot \det(D)$

$$\begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix} \cdot \begin{pmatrix} I_n & 0 \\ 0 & D \end{pmatrix} \overset{\text{partie}}{\underset{1.}{=\!=}} \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \qquad \text{(calcul direct)}$$

$$\det\begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix} = \begin{vmatrix} A & 0 \\ C & \begin{smallmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{smallmatrix} \end{vmatrix} = \det(A)$$

$$\det\begin{pmatrix} I_n & 0 \\ 0 & D \end{pmatrix} = \begin{vmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ 0 & & & D \end{vmatrix} = \det(D)$$

Alors :

$$\det\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \det\begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix} \cdot \det\begin{pmatrix} I_n & 0 \\ 0 & D \end{pmatrix}$$

$$= \det(A) \cdot \det(D)$$

(b) $\det(A) \neq 0$

$$\begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix}\begin{pmatrix} I_n & A^{-1}B \\ 0 & D-CA^{-1}B \end{pmatrix} \overset{\text{par la partie 1.}}{=}$$

$$= \begin{pmatrix} A & B \\ C & \cancel{CA^{-1}B} + D - \cancel{CA^{-1}B} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(c) M.q. si $\det(A) \neq 0$, alors $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \cdot \det(D - CA^{-1}B)$

Si $AC = CA$, alors $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB)$

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix} \cdot \det\begin{pmatrix} I_n & A^{-1}B \\ 0 & D-CA^{-1}B \end{pmatrix} =$$

$$= \det(A) \cdot \det(D - CA^{-1}B) \quad \checkmark$$

Si $AC = CA$, alors $\det(A) \cdot \det(D - CA^{-1}B) = \det(AD - ACA^{-1}B)$

$$= \det(AD - C\underbrace{AA^{-1}}_{I_n}B) = \det(AD - CB) \quad \checkmark$$

(d)

$$\det\begin{array}{c} \quad\; A \qquad\quad B \\ \left(\begin{array}{cc|cc} 2 & 1 & 1 & 1 \\ 3 & 2 & 0 & 2 \\ \hline 2 & 0 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{array}\right) \\ \quad\; C \qquad\quad D \end{array}$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1 \neq 0 \quad \checkmark$$

$AC \overset{?}{=} CA$ On remarque que $C = 2 I_2$

$AC = 2A = CA$ $\checkmark$

Donc $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB) =$

$$= \det\left(\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\right) =$$

$$= \det\left(\begin{pmatrix} 4 & 4 \\ 7 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}\right) = \det\begin{pmatrix} 2 & 2 \\ 7 & 3 \end{pmatrix} = -8$$