Série 9.

1. Produit matriciel (produit lignes fois colonnes).

(a)

$$\underbrace{\begin{pmatrix}4&1\\2&3\\-1&2\end{pmatrix}}_{3\times 2}\cdot\underbrace{\begin{pmatrix}2&-1\\3&2\end{pmatrix}}_{2\times 2} = \underbrace{\begin{pmatrix}4\cdot2+1\cdot3 & 4(-1)+1\cdot2\\ 2\cdot2+3\cdot3 & 2(-1)+3\cdot2\\ -1\cdot2+2\cdot3 & (-1)(-1)+2\cdot2\end{pmatrix}}_{3\times 2} = \begin{pmatrix}11&-2\\13&4\\4&5\end{pmatrix}$$

Sont-elles de type compatible ?

(b)

$$\underbrace{\begin{pmatrix}2&-1\\3&2\end{pmatrix}}_{2\times 2}\cdot\underbrace{\begin{pmatrix}4&1\\2&3\\-1&2\end{pmatrix}}_{3\times 2}$$

$2\neq 3$, donc elles ne sont pas $\rightsquigarrow$ de type compatible

(c)

$$\underbrace{\underbrace{\begin{pmatrix}-i&4&-1\\i&5&-i\end{pmatrix}}_{2\times 3}\cdot\underbrace{\begin{pmatrix}4&3&2\\5&6&3\\3&5&2\end{pmatrix}}_{3\times 3}}_{2\times 3}\cdot\underbrace{\begin{pmatrix}3&-4&3\\1&-2&2\\-7&11&-9\end{pmatrix}}_{3\times 3} = 2\times 3$$

$$\begin{pmatrix}4&3&2\\5&6&3\\3&5&2\end{pmatrix}\cdot\begin{pmatrix}3&-4&3\\1&-2&2\\-7&11&-9\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = I_3$$

$$\begin{pmatrix}-i&4&-1\\i&5&-i\end{pmatrix}\cdot I_3 = \begin{pmatrix}-i&4&-1\\i&5&-i\end{pmatrix}$$

(d)

$$\underbrace{\begin{pmatrix} 0 & 2 & 1 \\ -7 & 5 & 0 \\ 1 & 2 & 1 \end{pmatrix}}_{3\times 3} \cdot \underbrace{\begin{pmatrix} 4 & 0 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}}_{3\times 2} \cdot \underbrace{\begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 1 & 0 & -1 \end{pmatrix}}_{2\times 4} = 3\times 4$$

$$\begin{pmatrix} 0 & 2 & 1 \\ -7 & 5 & 0 \\ 1 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -23 & -5 \\ 6 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ -23 & -5 \\ 6 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 4 & -2 \\ -5 & -28 & -46 & 28 \\ 0 & 6 & 12 & -6 \end{pmatrix}.$$

2.) $f : \mathbb{R}^3 \to \mathbb{R}^2 \qquad f(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3,\ -x_2 + x_3)$

$g : \mathbb{R}^2 \to \mathbb{R} \qquad g(x_1, x_2) = 3x_1 + x_2$

$h : \mathbb{R}^2 \to \mathbb{R}^3 \qquad h(x_1, x_2) = (x_1 + 2x_2,\ -4x_1 + x_2,\ x_2)$

(a) $A = M(f)$, $B = M(g)$, $C = M(h)$

Rappel: base canonique de $\mathbb{R}^3$: $\{ \underbrace{(1,0,0)}_{=: e_1}, \underbrace{(0,1,0)}_{=: e_2}, \underbrace{(0,0,1)}_{=: e_3} \}$

$f(1,0,0) = (1, 0)$

$f(0,1,0) = (2, -1)$

$f(0,0,1) = (3, 1)$

$$A = M(f) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{pmatrix}$$

Cela revient à mettre les coefficients de x_1 sur la première colonne de A, ... etc.

$$B = M(g) = \begin{pmatrix} 3 & 1 \end{pmatrix} \qquad C = M(h) = \begin{pmatrix} 1 & 2 \\ -4 & 1 \\ 0 & 1 \end{pmatrix}$$

(b) $M(f\circ h)$, $M(g\circ f)$ et $M(g\circ h)$

$$M(f\circ h) = M(f)\cdot M(h) = A\cdot C = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{pmatrix}}_{2\times 3} \underbrace{\begin{pmatrix} 1 & 2 \\ -4 & 1 \\ 0 & 1 \end{pmatrix}}_{3\times 2}$$

$$= \begin{pmatrix} -7 & 7 \\ 4 & 0 \end{pmatrix}$$

$$M(g\circ f) = M(g)\cdot M(f) = B\cdot A = \underbrace{\begin{pmatrix} 3 & 1 \end{pmatrix}}_{1\times 2} \cdot \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{pmatrix}}_{2\times 3} \quad = 1\times 3$$

$$= \begin{pmatrix} 3 & 5 & 10 \end{pmatrix}$$

$$M(g\circ h) = M(g)\cdot M(h) = B\cdot C = \underbrace{\begin{pmatrix} 3 & 1 \end{pmatrix}}_{1\times 2} \underbrace{\begin{pmatrix} 1 & 2 \\ -4 & 1 \\ 0 & 1 \end{pmatrix}}_{3\times 2}$$

elles ne sont pas compatibles

Remarque: $\cancel{g\circ h}$ $\qquad h: \mathbb{R}^2 \to \mathbb{R}^3$

$\qquad g: \mathbb{R}^2 \to \mathbb{R}$

(c) Image de $x = (3,2)$ par $\cancel{f}$, g, h, $f\circ h$ et $\cancel{g\circ f}$.

pas compatibles avec $x = (3,2) \in \mathbb{R}^2$

$$g(x) = M(g)\,x = Bx = \underbrace{(3, 1)}_{1\times 2}\cdot \underbrace{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}_{2\times 1} = 11 \qquad = 1\times 1$$

$$h(x) = M(h)\cdot x = Cx = \underbrace{\begin{pmatrix} 1 & 2 \\ -4 & 1 \\ 0 & 1 \end{pmatrix}}_{3\times 2} \cdot \underbrace{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}_{2\times 1} = \begin{pmatrix} 7 \\ -10 \\ 2 \end{pmatrix} \qquad = 3\times 1$$

$$(f\circ h)(x) = M(f\circ h)\cdot x = \underbrace{\begin{pmatrix} -7 & 7 \\ 4 & 0 \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}_{2\times 1} = \underbrace{\begin{pmatrix} -7 \\ 12 \end{pmatrix}}_{2\times 1}$$

$\blacksquare$

3.) L'image d'un vecteur.

$\mathcal{B} = \{1, x, x^2, x^3\}$ (base canonique de $\mathbb{R}_3[x]$)

$\mathcal{C} = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ (aussi une base $\mathbb{R}_3[x]$)

App. lin : $f: \begin{array}{l} \mathbb{R}_3[x] \to \mathbb{R}_3[x] \\ p(x) \mapsto q(x) := p(x+1) \end{array}$

(a) Pour $g(x) = 1+2x+4x^2+8x^3$, calculer $[f(g)]_{\mathcal{B}}$

$$\begin{aligned}(f(g))(x) &= g(x+1) = 1+2(x+1)+4(x+1)^2+8(x+1)^3 = \\ &= 1+2x+2+4(x^2+2x+1)+8(x^3+1+3x^2+3x) = \\ &= 1+2x+2+4x^2+8x+4+8x^3+8+24x^2+24x = \\ &= 15+34x+28x^2+8x^3\end{aligned}$$

Donc $[f(g)]_{\mathcal{B}} = \begin{pmatrix} 15 \\ 34 \\ 28 \\ 8 \end{pmatrix}$

(b) $[f(g)]_{\mathcal{C}}$

Il faut trouver les réels a, b, c, d t.q. on a l'égalité

$$\begin{aligned}(f(g))(x) &= 15+34x+28x^2+8x^3 \stackrel{!}{=} \\ &= a\cdot(1)+b(1+x)+c(1+x+x^2)+d(1+x+x^2+x^3) \\ &= (a+b+c+d)+(b+c+d)x+(c+d)x^2+dx^3\end{aligned}$$

$$\rightsquigarrow \begin{cases} a+b+c+d = 15 \\ b+c+d = 34 \\ c+d = 28 \\ d = 8 \end{cases} \iff \begin{cases} a = -19 \\ b = 6 \\ c = 20 \\ d = 8 \end{cases}$$

$$[f(g)]_{\mathcal{C}} = \begin{pmatrix} -19 \\ 6 \\ 20 \\ 8 \end{pmatrix}$$

(c) Calculer de la matrice $M(f)_{\mathcal{B},\mathcal{C}}$.

Soit $\mathcal{B} = \{b_1, b_2, b_3, b_4\}$

$$M(f)_{\mathcal{B},\mathcal{C}} = \left[\begin{array}{c|c|c|c} [f(b_1)]_{\mathcal{C}} & [f(b_2)]_{\mathcal{C}} & [f(b_3)]_{\mathcal{C}} & [f(b_4)]_{\mathcal{C}} \end{array} \right]$$

On calcule les images des vecteurs de la base $\mathcal{B}$.

$\left(\begin{array}{l} \mathcal{C} = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\} \\ f : \text{« } x \text{ est remplacé par } x+1 \text{ »} \end{array} \right)$

$f(1) = 1 \cdot 1$

$f(x) = x+1 = 1 \cdot (1+x)$

$f(x^2) = (x+1)^2 = x^2+2x+1 = (1+x+x^2)+x$

$= (1+x+x^2)+(1+x)-1 = 1 \cdot (1+x+x^2) + 1 \cdot (1+x) - 1 \cdot 1$

$f(x^3) = (x+1)^3 = x^3+3x^2+3x+1 = (1+x+x^2+x^3)+2x^2+2x$

$= (1+x+x^2+x^3)+2(x+x^2)$

$= 1 \cdot (1+x+x^2+x^3) + 2(1+x+x^2) - 2 \cdot 1$

$$M(f)_{\mathcal{B},\mathcal{C}} = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(d) Calculer $[f(g)]_{\mathcal{C}}$. (Confirmer le résultat du pt (b))

Par Prop. 3.18, on a : $[f(g)]_{\mathcal{C}} = M(f)_{\mathcal{B},\mathcal{C}} \cdot [g]_{\mathcal{B}}$

$g(x) = 1 + 2x + 4x^2 + 8x^3$

$$[g]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix}$$

$$[f(g)]_{\mathcal{C}} = M(f)_{\mathcal{B},\mathcal{C}} \cdot [g]_{\mathcal{B}} = \underbrace{\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{4\times 4} \cdot \underbrace{\begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix}}_{4\times 1}$$

$$= \begin{pmatrix} -19 \\ 6 \\ 20 \\ 8 \end{pmatrix} \quad \checkmark$$

$\blacksquare$

4. Matrices Inversibles.

(a) $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$ et $y = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$(I_2 - A)^{-1}$? Résoudre $x - Ax - y = 0$.

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$I_2 - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

On calcule l'inverse de $(I_2 - A)$:

$$\left(I_2 - A \;\middle|\; I_2 \right) = \left(\begin{array}{cc|cc} 0 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right)$$

Échange 1e <-> 2e ligne :

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{array}\right) \xrightarrow{L_2' \leftarrow L_2 \cdot \left(-\frac{1}{2}\right)} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 \end{array}\right)$$

$$\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix} = (I_2 - A)^{-1}$$

$$(I_2 - A)^{-1} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

Maintenant : $\underbrace{x - Ax - y}_{} = 0 \qquad y = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Remarque : $Ix = x$

$$Ix - Ax - y = 0$$

$$(I - A)x = y$$

$$\underbrace{(I-A)^{-1}(I-A)}_{I_2} x = (I-A)^{-1} y$$

$$\rightsquigarrow \quad x = (I-A)^{-1} y = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

∎