ENRICO LAENG.

PARTIAL DIFFERENTIAL EQUATIONS.

Book: Arne Broman "Introduction to P.D.E."
From Fourier Series to Boundary-value problems; Dover.

Exam: English / Italian
"Oral" - Start with a topic of your choice (out of ~ 10 topics), plus some questions on other parts of the syllabus.

2 weeks from today outing on Vajont, probably class is cancelled, we'll make up for lost time.

O.D.E.

Ordinary differential equations:

e.g. 1) $y' = y$ $\rightarrow$ the unknown function is $y = y(x)$
$\rightarrow$ the solution is $y(x) = c \cdot e^{x}$ with $c \in \mathbb{R}$

2) $y'' + y' + y = \sin x$ we'll expect two constants.

P.D.E.

e.g. $\boxed{u_{xx} + u_{yy} = 0}$ the unknown is $z = u(x;y)$ function of $n \geq 2$

example of solution: $u(x;y) = x^2 - y^2$ (OUT OF A HUGE SPACE of HARMONIC FUNCTIONS)

check: $u_x = \frac{\partial}{\partial x} u = 2x$ $\quad u_y = -2y$

$u_{xx} = 2$ $\quad u_{yy} = -2$

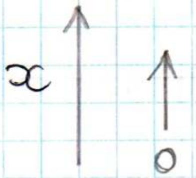
Basically arbitrary functions can appear in the solutions.

We use them to solve also physics problems.

⇒ The general solution of an ODE depends on n arbitrary constants, <u>BUT</u> if we attach to our ODE some suitable initial conditions, <u>THEN</u> the solution is unique.

⇒ A similar thing happens for PDE, but now there are arbitrary functions in the solution. If we attach some suitable boundary conditions, the solution becomes unique.

<u>eg.</u> Initial conditions:



$x(t)$

$\left.\begin{array}{l} x(0) \\ x'(0) \end{array}\right\}$ I.C.

What is PDE? <u>eg.</u> waving string; we have to give:

→ initial displacement of string

→ initial velocity

In this course we'll study PDE using some mathematical TOOLS:

- Fourier Series (ordinary & generalized)
- " Transform
- Laplace "
- Separation of variables
- Other miscellaneous methods.

QUICK REVIEW OF SERIES OF NUMBERS.

(Let's review numerical series to start with, then we'll switch to series of functions).

$$\boxed{\sum_{k=0}^{\infty} a_k} \quad (*)$$

$k=0$ ← index

The series (*) **CONVERGES** if $\lim_{N\to\infty} \sum_{k=0}^{N} a_k$ exists and is <u>finite</u>.

$\sum_{k=0}^{N} a_k$ → N-th partial sums

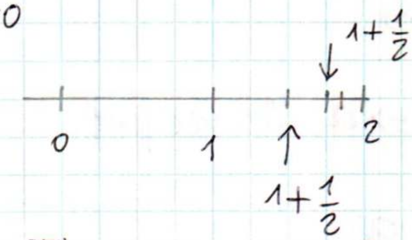
<u>Theorem</u>: <u>Necessary</u> (but <u>not sufficient</u>) condition for (*) to converge is $\boxed{\lim_{k\to\infty} a_k = 0}$

If the $\lim_{N\to\infty} \sum_{k=0}^{N} a_k$ is $+\infty$ or $-\infty$ we say that the series **DIVERGES**. If this limit does not exist the series is **INDETERMINATE**.

- Geometric Series:

$$\sum_{k=0}^{\infty} \alpha^k = 1+\alpha+\alpha^2+\alpha^3+\ldots$$

INTERESTING if $\alpha<1$, then the series converges.



If $\alpha = \frac{1}{2}$ the series converges to 2.

<u>Th.</u> The Geometric Series $\sum_{k=0}^{\infty} \alpha^k$ converges if $|\alpha|<1$ ($\alpha \in \mathbb{R}$, actually it works also for $\alpha \in \mathbb{C}$) and its sum is exactly $\frac{1}{1-\alpha}$.

<u>Proof</u>: $S_N = \sum_{k=0}^{N} \alpha^k = 1+\alpha+\alpha^2+\ldots+\alpha^N$

$\alpha S_N = \cancel{\alpha}+\cancel{\alpha^2}+\alpha^3+\ldots+\alpha^{N+1}$

we subtract these terms obtaining

$S_N - \alpha S_N = 1-\alpha^{N+1}$

$(1-\alpha) S_N = 1-\alpha^{N+1}$ ← we collect S_N

$$S_N = \frac{1-\alpha^{N+1}}{1-\alpha}$$

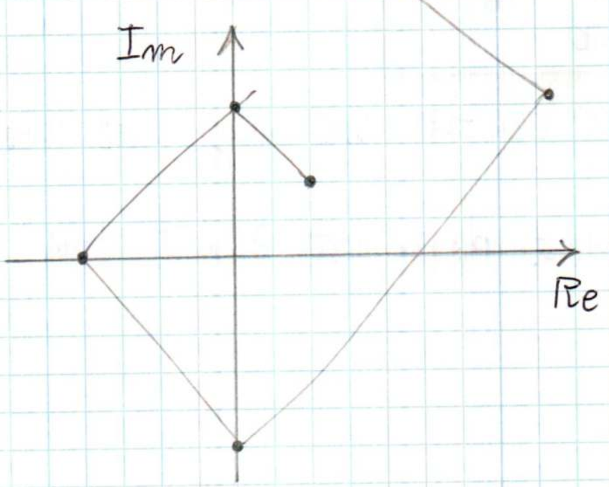
Now, if $|\alpha| < 1$, the $\lim_{N\to\infty} \alpha^N = 0$ So $\lim_{N\to\infty} S_N = \frac{1}{1-\alpha}$

Q.D.E. = Quod Erat Demonstrandum

eg. $\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n = 1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \dots = \frac{1}{1+\frac{2}{3}} = 3/5$

eg. $\sum_{n=0}^{\infty} \left(\frac{1+i}{5}\right)^n = \frac{1}{1-\frac{1+i}{5}} = \frac{5}{5-1-i} = \frac{5}{4-i} \cdot \frac{4+i}{4+i} = \frac{20+5i}{4^2+1} =$

$= \frac{20}{17} + \frac{5}{17} i$



Graphic representation of a geometric series when $\alpha \in \mathbb{C}$.

For $|\alpha| \geq 1$ we get, e.g. (for example),
EXEMPLI GRATIA

$\alpha = 1 \quad \sum_{n=0}^{\infty} 1^n = 1+1+1+1+1+\dots \to \infty$ Divergent

$\alpha = 3 \quad \sum_{n=0}^{\infty} 3^n = 1+3+9+\dots \longrightarrow \infty$ "

$\alpha = -2 \qquad 1-2+4-8+16\dots$??? INDETERMINATE

- Another example: HARMONIC SERIES

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$, DIVERGENT to $+\infty$ despite the fact $\lim_{n\to\infty} \frac{1}{n} = 0$ [N.C. (Necessary Condition)]

(this is a case where the Necessary-but-not-sufficient condition for the series to converge is satisfied, but the series doesn't converge)

We'll review a few theorems called CONVERGENCE TESTS FOR SERIES. These are SUFFICIENT (not NECESSARY) conditions for convergence. In many cases we can prove with some suitable conv. test that a series converges, but we have no exact formula for its sum.

Ratio test Given $\sum_{n=0}^{\infty} a_n$ we look at

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \ell$$

- if $\ell < 1$ then series CONVERGES
- if $\ell > 1$ does not CONVERGE
- if $\ell = 1$ the test fails (DOES NOT MEAN THE SERIES DOESN'T CONVERGE)

ex 1) $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$

(Remember $0! = 1$)

Ratio test $a_n = 1/n!$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1} \to 0 \to \text{SO CONVERGES}$$

In particularly it converges to $e \cong 2{,}718\dots$ IT IS THE TAYLOR SERIES OF THE EXP.

ex 2) $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$

Ratio test: $\lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim \frac{n^2}{(n+1)^2} = \lim \left(\frac{n}{n+1}\right)^2 = 1$

The test fails (because the limit tends to 1)

DO NOT conclude that convergence fails!!

> We'll see later, using a different test, that this series converges. We'll see much later, using Fourier Series, that the sum is $\pi^2/6$ (π-squared-over-six).

SEE PAGE II OF TUTORIAL 1 AND ALSO P. 34R

Root Test

Given $\sum_{\kappa=0}^{\infty} a_\kappa$ we look at

$$\lim_{\kappa\to\infty} |a_\kappa|^{1/\kappa} = \ell \quad \begin{cases} \ell<1 & \text{converges} \\ \ell>1 & \text{not converge} \\ \ell=1 & \text{test fails} \end{cases}$$

(Direct) Comparison test. Given two series $\sum_{\kappa=0}^{\infty} a_\kappa$ & $\sum_{\kappa=0}^{\infty} b_\kappa$

(let's assume $a_\kappa > 0$ & $b_\kappa > 0$ for simplicity) $(c>0)$

- If $a_\kappa \leq c\cdot b_\kappa$ for $\kappa = \kappa_0, \kappa_0+1, \kappa_0+2, \dots$ and $\sum b_\kappa$ converges then $\sum a_\kappa$ converges.
- If $a_\kappa \geq c b_\kappa$ for $\kappa = \kappa_0, \kappa_0+1, \kappa_0+2, \dots$ and $\sum b_\kappa$ diverges then $\sum a_\kappa$ diverges.

e.g. $\sum_{\kappa=1}^{\infty} \frac{1}{\sqrt{\kappa}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots$

Note that $\frac{1}{\sqrt{\kappa}} > \frac{1}{\kappa}$ for $\kappa = 2, 3, \dots$ $(\kappa > \sqrt{\kappa})$

so by comparison test (case 2) with $\sum_{\kappa=1}^{\infty} 1/\kappa$ (Page 2R Bottom)
(divergent H.S.) it diverges.

Limit Comparison test

Consider $\sum a_\kappa$ and $\sum b_\kappa$ and

$$\lim_{\kappa\to\infty} \left|\frac{a_\kappa}{b_\kappa}\right| = \alpha \quad \text{with} \quad \boxed{0<\alpha<+\infty}$$

(I.F. $0/0$) Then either both series converge or both series do not converge.

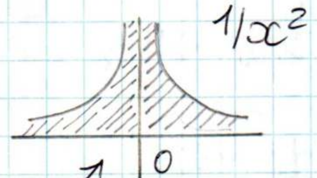
Because from the Necessary But Not Sufficient condition we must have $a_\kappa, b_\kappa \to 0$ as $\kappa\to\infty$

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ if f is continuous in $[a;b]$ and $F'(x) = f(x)$

Generalized integrals

ex. $\int_{-1}^{1} 1/x^2 \, dx = \left[-1/x\right]_{-1}^{1} = -1-1 = -2$?

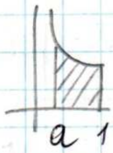


it isn't continuous

The true result must be obtained as a limit:

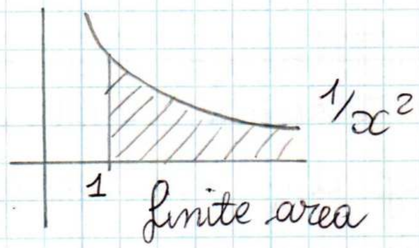
$$\lim_{a \to 0^+} \int_{-1}^{-a} x^{-2} \, dx + \int_{a}^{1} x^{-2} \, dx = \dots = +\infty$$

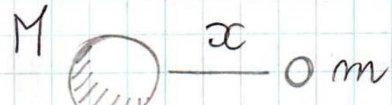
ex. $\int_0^1 x^{-1/2} \, dx \stackrel{\text{def.}}{=} \lim_{a \to 0^+} \int_a^1 x^{-1/2} \, dx =$



$$= \lim_{a \to 0^+} \left(2\sqrt{1} - 2\sqrt{a}\right) = 2$$

ex. $\int_1^{+\infty} x^{-2} \, dx \stackrel{\text{def}}{=} \lim_{b \to +\infty} \int_1^b x^{-2} \, dx = \lim_{b \to +\infty} \left(-\frac{1}{b} + 1\right) = 1$

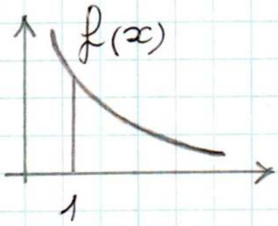




$$F = \varkappa \frac{Mm}{x^2}$$

Energy associated can be calculated as an improper integral.

Integral comparison Test.



Assume $f(x)$ continuous on $[1; +\infty]$

$f(x) \geq 0$ $f(x)$ decreasing $\lim_{x \to +\infty} f(x) = 0$

Let's define $a_n = f(n)$ with $n = 1, 2, 3 \dots$ (samples of f at the integers)

then the improper integral $\int_1^{+\infty} f(x) \, dx$ and the series $\sum_{n=1}^{\infty} a_n$ <u>either</u> <u>both</u> converge <u>or</u> <u>both</u> diverge to $+\infty$.

<u>N.B.</u>: the starting point $n=1$ could also be $n=2$ or other values.

<u>N.B.</u>^{2} If both converge <u>we are not</u> saying that the sum of the series coincides with the value of the integral.

We saw before that the ratio test fails with $\sum 1/n^2$
Now, take $f(x) = \frac{1}{x^2}$ we have $\int_1^{+\infty} x^{-2}\,dx = 1$
$a_n = f(n) = 1/n^2$ so, by the I.C.T., the series $\sum_{n=1}^{\infty} 1/n^2$ CONVERGES. (the sum is $\pi^2/6$) (F.34R)
This is a more powerful test.

<u>Remark</u> In all the conv. tests reviewed so far, we never used the signs of a_n, we only used $|a_n|$. There are cases in which series converges because of cancellations of + and − signs.

<u>Leibnitz (alternating series) test.</u>

Suppose $a_n = (-1)^n b_n$, with $b_n > 0$;
look at the series $\sum_{n=0}^{\infty} a_n = b_0 - b_1 + b_2 - b_3 + \dots$

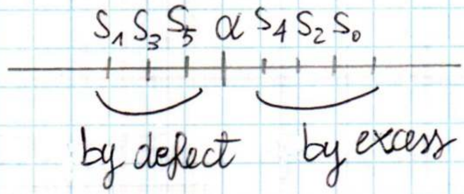
<u>if</u> 1) $b_n > 0$

2) $\lim_{n\to\infty} b_n = 0$ (N.C. for convergence)

3) $b_{n+1} < b_n$ for $n = 0, 1, 2, 3, \dots$

<u>then</u> our series (with alternating signs) converges.
<u>Furthermore</u>, if we look at its partial sums, these

are greater than the sum of the series α if the last term is positive, smaller than α if the last term is negative.



ex.) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

$$a_n = (-1)^n \frac{1}{n+1} \qquad b_n = \frac{1}{n+1}$$

(1) $\frac{1}{n+1} > 0$ for $n = 0, 1, 2, \dots$

(2) $\displaystyle\lim_{n\to\infty} \frac{1}{n+1} = 0$ (3) $b_n = \frac{1}{n+1} > \frac{1}{n+2} = b_{n+1}$

$\Longrightarrow$ the series CONVERGES.

(we will see that the sum is $\log 2 \cong 0{,}69\dots$)

[N.B. $\log x = \log_e x$ IN EVERY SERIOUS COURSE of ANALYSIS!]

$S_0 = 1 > 0{,}69\dots$ $\quad S_1 = 1 - \frac{1}{2} = 0{,}5 < 0{,}69\dots$

$S_2 = 1 - \frac{1}{2} + \frac{1}{3} = 0{,}5 + 0{,}333 = 0{,}833\dots > 0{,}69$

$S_3 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} < 0{,}69\dots$

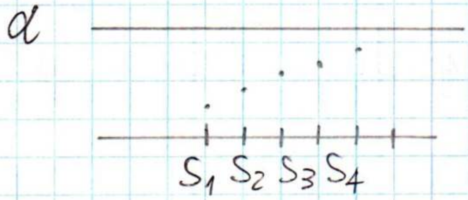
of course a series with terms of variable sign could be more complicated (eg. $++---++-\dots$), so there are more refined tests...

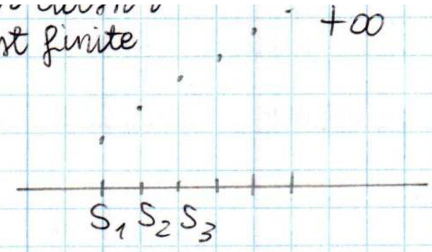
Actually, the terms a_n could be in $\mathbb{C}$.

Remark If the terms of the series are all >0, then its partial sums $S_N = \sum_{n=0}^{N} a_n$ are monotonic increasing in N. $\Rightarrow$ Either $\lim_{N\to\infty} S_N$ exists finite (the series converges) or $\lim_{N\to\infty} S_N = +\infty$ (diverges).

$\boxed{\text{Indeter. series NO.}}$ THIS IS THE IMPORTANT THING.

The limit exists finite ... ... does not exist finite





Def. We say that $\sum_{x=0}^{\infty} a_x$ is ABSOLUTELY convergent if the other series $\sum_{x=0}^{\infty} |a_x|$ converges.

Def. If $\sum_{x=0}^{\infty} a_x$ converges BUT $\sum_{x=0}^{\infty} |a_x| = +\infty$ we say that ↑ is CONDITIONALLY convergent.

ex. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \log 2$ is conditionally convergent because $\sum_{x=0}^{\infty} |a_x| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = +\infty$

ex. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots$ this is absolutely convergent.

Th. the commutative property of sums is TRUE for ABSOLUTELY convergent series, is FALSE for CONDITIONALLY convergent series.

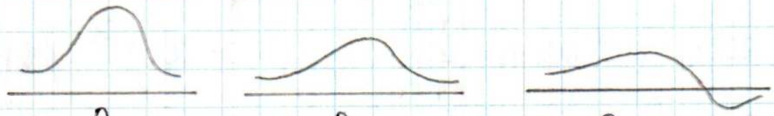
in particular: $\sum_{x=1}^{+\infty} 1/x^2$ converges for the Integral Comp. Test;

$\sum_{x=1}^{+\infty} (-1)^{x+1} \cdot 1/x^2$ converges for the Leibnitz test.

SERIES OF FUNCTIONS

Introduction.

$\sum_{n=0}^{\infty} f_n(x)$



$f_0(x)$ $f_1(x)$ $f_2(x)$

$\dots$ $f_N(x)$

(*) <u>Def.</u> The series $\sum_{n=0}^{\infty} f_n(x)$ <u>converges pointwise</u>, if for every $x \in A \subseteq \mathbb{R}$, every fixed $x \in A$ the numerical series $\sum_{n=0}^{\infty} \alpha_n$ converges (where $\alpha_n = f_n(x)$)

<u>Necessary condition</u> for convergence is $\boxed{\lim_{n \to \infty} f_n(x) = 0}$ for $\forall x \in A$.

The previous definition (*) is equivalent to

$\boxed{\lim_{N \to \infty} \sum_{n=0}^{N} f_n(x) = F(x)}$ for $\forall x \in A$.

<u>N.B.</u> if we write x in the G.S. $\sum_{n=0}^{\infty} x^n$ and choose $A = (-1; 1) \subset \mathbb{R}$ we have <u>proven</u> that in A the G.S. converges <u>pointwise</u> to $F(x) = \frac{1}{1-x}$.

Actually the G.S. is a special case of <u>power series</u> $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ where $c_n = 1$ and $x_0 = 0$

We'll see that power series behave a bit like polynomials.

for example from $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ get

$\sum_{n=0}^{\infty} n x^{n-1} = \left(\frac{1}{1-x}\right)' = \left[(1-x)^{-1}\right]'$ (that is the 1^{st} derivative)

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

true for $x \in (-1; 1)$

Connection between $\sum_{n=0}^{\infty} c_n (x-x_0)^n$ (*) power series (<u>Taylor series</u>) and the function $f(x)$, sum of this series on some $A \subset \mathbb{R}$.

<u>Th. 1)</u> To any P.S. (*) we can associate a number $R \geq 0$ (possibly $R = +\infty$) RADIUS of CONVERGENCE such that if $x \in (x_0 - R; x_0 + R)$ then the P.S. <u>converges absolutely.</u>

If $|x - x_0| > R$ (i.e. $x \notin (x_0 - R; x_0 + R)$) then the P.S. does not converge.

If $x = x_0 + R$ or $x = x_0 - R$ this must be studied case by case.

<u>Th. 2)</u> If we called $f(x)$ the sum of our P.S. for $x \in (x_0 - R; x_0 + R)$ then $\boxed{c_n = \frac{f^{(n)}(x_0)}{n!}}$

<u>ex.1</u> $f(x) = e^x \quad f'(x) = e^x \quad f''(x) = e^x \quad \dots$

$$x_0 = 0 \qquad f(0) = e^0 = 1 \qquad f'(0) = f''(0) = \dots = 1$$

(Mac Laurin Series) $$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

<u>Claim</u> $R = \infty$

<u>Proof</u> of claim using RATIO TEST

{ Actually sometimes is possible to calculate the radius of convergen with the ratio test.

$$\sum_{n=0}^{\infty} a_n \qquad a_n = \frac{x^n}{n!}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \frac{n!}{(n+1)!} |x| = \frac{1}{n+1} |x|$$

So, for any fixed $x \in \mathbb{R}$, as $n \to \infty$ we get

$\lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$ by the ratio test our series

converges $\forall x \in \mathbb{R} \Rightarrow R = \infty$

Special case: $e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \ldots$

With a very simple computation we can show that

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ (odd function)

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ (even function)

Euler's Formula:

$$\boxed{e^{x+iy} = e^x(\cos y + i \sin y)}$$

Proof: let's take for granted $e^{x+iy} = e^x \cdot \boxed{e^{iy}}$ we concentrate on this term

Use $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots$ $(R = \infty)$

with $t = iy$: [Remember $i^2 = -1$]

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \ldots =$$

$$= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right)$$

(We collected separately real & imaginary parts)

$= \cos y + i \sin y$ Q.E.D.

Corollaries: $\boxed{e^{\pi i} = -1}$ $\quad e^{\pi i} + 1 = 0$

$$\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}} \qquad \boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2}}$$

LESS.2 09/10/2008

Every series can be written as:

$$\sum_{k=0}^{\infty} a_k = \underbrace{\sum_{k=0}^{N} a_k}_{S_N} + \underbrace{\sum_{k=N+1}^{\infty} a_k}_{R_N}$$

SERIES can be split into two parts:

S_N: N-th partial sum (FINITE SUM)

R_N: N-th remainder (ANOTHER SERIES)

$$\sum_{k=0}^{\infty} a_k \text{ CONVERGES} \iff \boxed{\lim_{N\to\infty} R_N = 0}$$ (NECESSARY & SUFFICIENT)

NECESSARY COND.: $\lim_{k\to\infty} a_k = 0$

The convergence tests (ratio, root, comparison, limit comparison, $\int$ test, Leibnitz, ...) are SUFFICIENT conditions for convergence.

Sometimes it is possible to find estimates of the kind

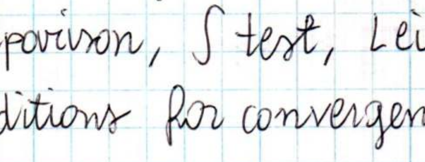
$$\boxed{|R_N| < g(N) \to 0 \quad \text{as } N \to \infty}$$

Useful in practice because if $g(N) < 10^{-h}$ (h integer) we know that h decimal digits should be OK.

e.g. $\sum_{k=0}^{\infty} (-1)^k b_k$ (*) Leibnitz: (1) $b_k \to 0$ as $k \to \infty$ (N.C.)

(2) $b_{k+1} < b_k$

(3) $b_k \geq 0$



then $|R_N| < b_{N+1}$

In other cases it might be more tricky to estimate $|R_N|$ but we will see some examples...