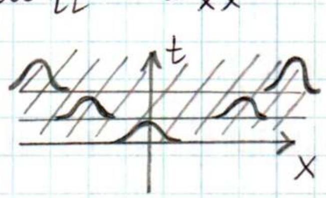
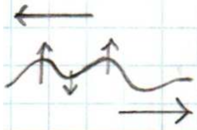
$$\begin{cases} u_{tt} = u_{xx} & \text{unknown function } u \equiv u(x,t) \\ & \Omega \equiv \{x \in \mathbb{R},\, t > 0\} \\ \text{INITIAL POSITION} & u(x,0) = u_0(x) \quad \text{known function} \\ \text{INITIAL VELOCITY} & u_t(x,0) = u_1(x) \quad \text{" \quad "} \end{cases}$$





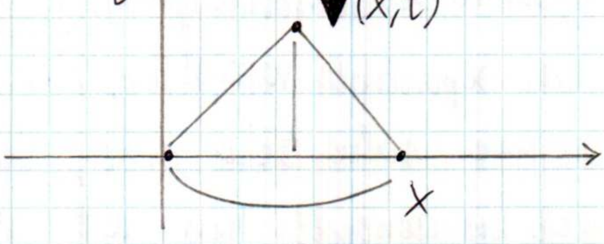
Solution (d'Alambert formula) [57R]

$$u(x,t) = \frac{1}{2}\left[u_0(x+t) + u_0(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} u_1(\xi)\, d\xi \quad \text{(done before)}$$

This problem is a highly idealized model of a physical "vibrating string" (∞ lenght along x). Today we'll study a more realistic model where the string has finite lenght and it's fixed at the 2 boundary points.

N.B. The problem is well posed if
1) the solution is unique;
2) a small perturbation of $u_1(x)$ and $u_0(x)$ produces a small perturbation of $u(x,t)$.

Remark in this solution for the ∞ string problem we find 2 waves travelling in opposite directions along x (they interphere with each other), also there is a third term that depends on the integral of the initial velocity. The value of $u(x,t)$ only depends on the boundary data $u_0(x)$ and $u_1(x)$ on the interval $[x-t; x+t]$, that becomes bigger & bigger as time flows.

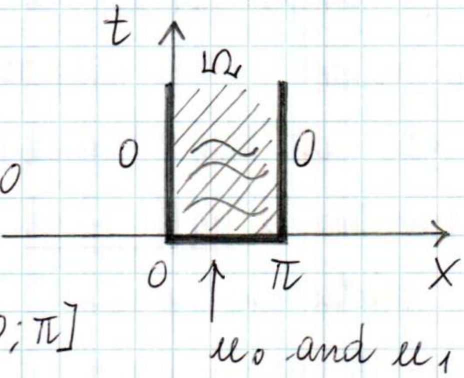


this picture, and the "interval of influence" $[x-t; x+t]$ which grows with time t, are related to the fact that there is a velocity of propagation (hyperbolic/wave eqq.)

In this model, there is no friction (the energy of the string is in this model equal to kinetic energy + potential energy, this total energy $E(t)$ is CONSTANT in time).
For macroscopic physical strings (e.g. guitar) a more realistic model should include friction (part of the macroscopic cinetic energy of the string gets transformed into microscopic cinetic energy (heat)).

[142] Same PDE for finite string (to keep the formulas simple, without extra parameters, we choose π for the lenght of the string, but it is easy to solve the same problem with any lenght).

$$u_{tt} = u_{xx}$$
$$u(0,t) = u(\pi,t) = 0 \quad \forall t > 0$$
$$u(X,0) = u_0(X)$$
known func. for $X \in [0;\pi]$
$$u_t(X,0) = u_1(X)$$



Before we "do the math" let us keep in mind that in the solution of this problem there will also be 2 waves travelling in opposite directions BUT they will reflect at the two endpoints $x=0$, $x=\pi$ and will interphere with each other in a more complicated fashion. Also the third term (depending on u_1) will reflect at the endpoints. These reflections are ODD

The idea (goes back to Fourier) is to start solving an easier problem where our solution has separable variables, i.e., $v(x,t) = \mathcal{X}(x)\, T(t)$. If we can solve this case, the full solution $u(x,t)$, which in general has not separable variables, will be a series of solution of the easier problem.

Plug $u = v(x,t) = X(x)\,T(t)$ into the PDE $u_{tt} = u_{xx}$

$$X(x)\,T''(t) = X''(x)\,T(t)$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = -\lambda$$

(we expect sin/cos with frequency s.t. the endpoints are fixed; **not** real exponential)

This peculiar equality is possible only if the two ratios are equal to the same constant.

ODE $X''(x) + \lambda X(x) = 0$ $\qquad y^2 + \lambda = 0 \qquad y^2 = -\lambda$

ODE $T''(t) + \lambda T(t) = 0$ $\qquad y = \pm\sqrt{\lambda}\, i$

$$X(x) = C_1 \cos\sqrt{\lambda}\, x + C_2 \sin\sqrt{\lambda}\, x \quad \text{(with Euler's formula)}$$

We must have $X(0) = X(\pi) = 0$ because of the boundary conditions (also we want a finite number of zeros of $X(x)$ in the interval $x \in [0;\pi]$); from $X(0)=0$ we get $C_1 = 0$.

We must have $\sqrt{\lambda} = n$ integer $\Rightarrow \lambda = n^2$ (we have to assume this if we want to satisfy $X(\pi) = 0$)

Since λ is the same in both ODE's we have

$$T''(t) + n^2 T(t) = 0$$

$$\Rightarrow v(x,t) = \underbrace{(a_n \cos nt + b_n \sin nt)}_{T(t)}\, \underbrace{\sin nx}_{X(x)}$$

here we let the nonzero constant C_2 be absorbed into the constant a_n and b_n

We have shown that these for $n = 1, 2, 3, \ldots$ and for suitable coeff's a_n and b_n are separable variable solutions.

We have not used yet $u_0(x)$ and $u_1(x)$.

Let's try (Fourier) a solution of this form:

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \sin nx \qquad (*)$$

choosing the (∞ many) coeff's a_n, b_n in such a way that the B.C. for u_0 and u_1 are satisfied; a_n and b_n are determined by expanding u_0 and u_1 in F.S. ...

N.B. each term of (*) (for some n fixed) is a function with separable variables BUT the sum of the series $u(x,t)$ IS NOT.

$$\left.\begin{aligned} u(x,0) &= u_0(x) = \sum_{n=1}^{\infty} a_n \sin nx \\ u_t(x,0) &= u_1(x) = \sum_{n=1}^{\infty} n b_n \sin nx \end{aligned}\right\rangle x \in [0;\pi]$$

P. 144 BROMAN
P. 157 FARLOW

Let $\tilde{u}_0(x)$ be the ODD 2π-periodic extension of $u_0(x)$ from $x \in [0;\pi]$ to $x \in (-\infty;+\infty)$

Let $\tilde{u}_1(x)$ be the ODD 2π-periodic extension of $u_1(x)$ from $x \in [0;\pi]$ to $x \in (-\infty;+\infty)$

It makes sense to solve the ∞-string problem with periodic boundary data $\tilde{u}_0(x)$ and $\tilde{u}_1(x)$; we know its solution (d'Alambert):

$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{u}_0(x+t) + \tilde{u}_0(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{u}_1(\xi)\,d\xi$$

$$= \sum_{n=1}^{\infty} a_n \frac{1}{2}\left[\sin n(x+t) + \sin n(x-t)\right] - \sum_{n=1}^{\infty} b_n \frac{1}{2}\left[\cos n(x+t) - \cos n(x-t)\right]$$

by trigon. properties

$$= \sum_{n=1}^{\infty} \left(a_n \underbrace{\sin nx}_{\text{collect}} \cos nt + b_n \underbrace{\sin nx}_{\text{collect}} \sin nt\right) \quad \text{which is} \equiv (*)$$

(the solution of the ∞ string coincides with the one of the finite string)

We see that the method of Fourier produces this series solution which coincides with the solution of an auxiliary problem on the line restricted to $[0;\pi]$. The ODD reflections (fitting with the physics) are embedded in this construction.

Suppose we know that the solution of the problem for the ∞ string Ω is unique. One could object that, even though the restriction of this solution to $[0;\pi]$ with B.C. $\tilde{u}_0, \tilde{u}_1$, is a solution, perhaps this solution is not unique. This is NOT the case, but if we want to be rigorous a proof is required.

To prove <u>uniqueness</u> (and for other reasons...) we introduce the Energy $E(t)$ of our vibrating on $[0;\pi]$

$$E(t) = \int_0^\pi [\underbrace{u_t^2(x,t)}_{\text{cinetic energy}} + \underbrace{u_x^2(x,t)}_{\text{potential energy}}]\, dx$$

<u>Theorem</u> $E(t)$ is constant in time (i.e. if the cinetic energy decreases, the potential energy increases (compensates it) and viceversa, also there is no friction, i.e., transformation of macroscopic cinetic energy into microscopic (heat)).

<u>Proof:</u>
$$\frac{dE}{dt} = \int_0^\pi \frac{d}{dt}\left(u_t^2(x,t) + u_x^2(x,t)\right) dx =$$

$$= \int_0^\pi \left(2u_t(x,t)\frac{\partial}{\partial t}u_t(x,t) + 2u_x(x,t)\frac{\partial}{\partial t}u_x(x,t)\right) dx =$$

$$= \int_0^\pi 2\left(u_t \cdot \underset{\overset{\text{PDE}}{|||}}{u_{tt}} + u_x \cdot u_{xt}\right) dx =$$

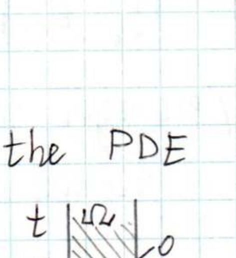
$$= \int_0^\pi 2\left(u_t \cdot u_{xx} + u_x \cdot u_{xt}\right) dx =$$

$$= 2\int_0^\pi \frac{\partial}{\partial x}(u_t u_x)\, dx \overset{\text{FTC}}{=} 2[u_t u_x]\Big|_{x=0}^{x=\pi} =$$

$= 0$ because $u_t = 0$ when $x = 0$ or $x = \pi$
(the velocity where the string is attached is 0)

Q.E.D.

<u>Proof of uniqueness</u>:
Suppose $v(x,t)$ and $w(x,t)$ are two solutions of the PDE problem for the vibrating string on $[0;\pi]$,



then the function $u(x,t) = v(x,t) - w(x,t)$ satisfies $u_{tt} = u_{xx}$ in Ω

$u(0,t) = u(\pi,t) = 0 \quad \forall t > 0$

$u(x,0) = 0$ for $x \in [0;\pi]$ $\quad u_t(x,0) = 0$ for $x \in [0;\pi]$

this $u(x,t)$ has $E(0)=0$ and, by the previous theorem, $E(t)=0 \quad \forall t>0$

$$u^2_{t(x,t)} + u^2_{x(x,t)} = 0$$

this is a sum of 2 positive addends; in order for it to be 0 both addends must be zero and this implies $u_x = 0$ for $\forall (x,t)$ and $u_t = 0$ for $\forall (x,t)$.

(in order for $\int_0^\pi (u_t^2 + u_x^2)\,dx = 0$ since the function we are integrating is ≥ 0 ...)

$$\Rightarrow \boxed{\underbrace{u(x,t)}_{\text{FCT}} = \underbrace{u(0,t)}_{0 \text{ because of B.C.}} + \int_0^x \underbrace{\frac{\partial u}{\partial x}(\xi,t)}_{0 \text{ because of}}\,d\xi \equiv 0} \qquad (t \text{ FROZEN})$$

Finally $\Rightarrow v(x,t) - w(x,t) = 0$ the 2 sol's are the same!

If you look at page 146 in the book it shows, using $E(t)$, that the problem is stable. We do it in a different way:

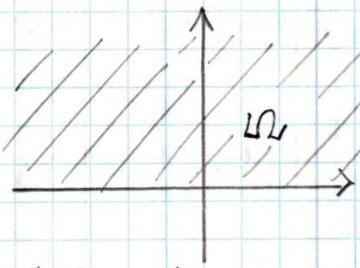
(1) we show that the PDE problem for the ∞-string is stable

(2) using the fact shown before, that the unique solution for the string on $[0;\pi]$ is the restriction of a problem for the ∞-string with B.C. $\tilde{u}_0(x)$ and $\tilde{u}_1(x)$.

Look at $u_{tt} = u_{xx}$

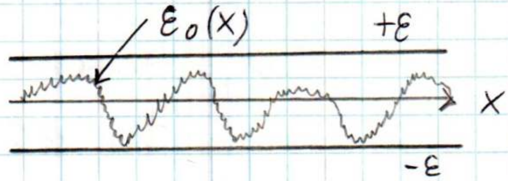
$u(x,0) = u_0(x) + \varepsilon_0(x)$

$u_t(x,0) = u_1(x) + \varepsilon_1(x)$



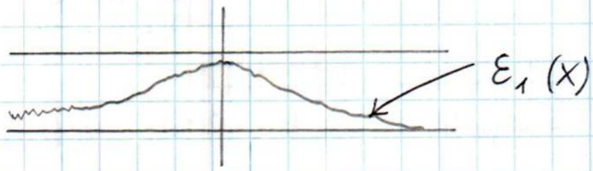
where $\varepsilon_0(x)$ and $\varepsilon_1(x)$ are "small" functions that perturb the original problem. Namely

(*) $|\varepsilon_0(x)| < \varepsilon \qquad \forall x \in \mathbb{R}$



(**) $\left|\int_0^x \varepsilon_1(\xi)\,d\xi\right| < \varepsilon \quad \forall x \in \mathbb{R}$

(enough decay as $x \to \pm\infty$ to satisfy (**))



Let's call $v(x,t)$ the solution of the perturbed problem and $u(x,t)$ the solution of the original problem

(with $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$)