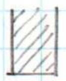
$$|v(x,t) - u(x,t)| = \Big| \tfrac{1}{2}\big(\cancel{u_0(x+t)} + \varepsilon_0(x+t) + \cancel{u_0(x-t)} + \varepsilon_0(x-t)\big) + \tfrac{1}{2}\int_{x-t}^{x+t} [\cancel{u_1(\xi)} + \varepsilon_1(\xi)]\, d\xi$$

$$- \tfrac{1}{2}\big(\cancel{u_0(x+t) + u_0(x-t)}\big) - \tfrac{1}{2}\int_{x-t}^{x+t} \cancel{u_1(\xi)}\, d\xi \Big| \leq \tfrac{1}{2}|\varepsilon_0(x+t) + \varepsilon_0(x-t)| +$$

$$+ \tfrac{1}{2}\Big|\int_{x-t}^{x+t} \varepsilon_1(\xi)\, d\xi\Big|$$

$< \frac{1}{2} 2\varepsilon + \frac{1}{2} 2\varepsilon = 2\varepsilon$ <u>arbitrary small</u>

Stability (proven now for ////////) is inherited by the other problem



$0 \quad \pi$

Let's say something new about <u>elliptic</u> PDE's comparing with what we have seen today.

<u>ex.</u>

$$(*)\begin{cases} u_{tt} + u_{xx} = 0 \quad \text{elliptic } (u_{tt} = -u_{xx}) \\ \text{this PDE alone means } \underline{\text{harmonic functions}} \\ u(x,0) = 0 \quad ////// \\ u_t(x,0) = \varepsilon \sin\frac{x}{\varepsilon} \end{cases}$$

(here there is a "smoothing" effect that <u>does not</u> hold for hyperbolic PDE's)

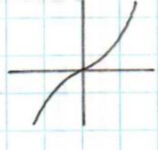
Unlike the previous apparently similar problem, this is <u>very</u> <u>unstable</u>.

We can find the solution $u(x,t) = \varepsilon^2 \sinh\left(\frac{t}{\varepsilon}\right) \sin\left(\frac{x}{\varepsilon}\right)$

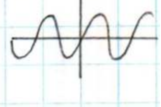
(at home check that this actually satisfies (*)
need to compute 2 second partial derivatives, and compute 2 limits as $t \to 0^+$)

We have $\text{Max } u(x,t) = \varepsilon^2 \sinh\left(\frac{t}{\varepsilon}\right)$
$\to \infty$ (quickly) as $\varepsilon \to 0$

$\sinh x = \dfrac{e^x - e^{-x}}{2}$



$\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

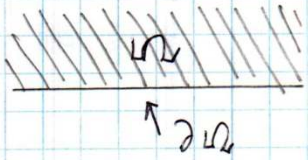


The same problem with 2^{nd} B.C. $u_t(x,0) = 0$
has solution $u(x,t) = 0 \xRightarrow{\text{IMPLIES}}$ a very small change in the 2^{nd} B.C. (from 0 to $\varepsilon \sin\frac{x}{\varepsilon}$) can produce a very large change

in the solution.

Let's remember a case (elliptic) solved a while ago via F.T. [P.55]

$$\begin{cases} u_{tt} + u_{xx} = 0 \\ u(x,0) = f(x) \end{cases}$$



NO B.C. on $u_t(x,0)$

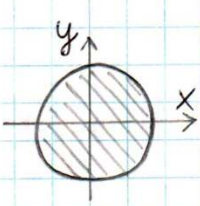
Solution $\displaystyle u(x,t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{(x-t)^2+y^2} f(t)\, dt$

This is a well posed problem!

> The bad behavior in the previous example was given by the B.C. on $u_t(x,0)$ which usually we <u>do not</u> include in an elliptic problem.

<u>There are</u> the so-called <u>Neumann problems</u> for elliptic PDE's where B.C. on u_t appear BUT they must satisfy some compatibility conditions (we'll not study it in this course).

[P.160] A similar (slightly more complicated) well-posed elliptic problem is



$\Omega \equiv \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$

$\partial\Omega \equiv \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } (x,y) \in \Omega \quad (u \text{ harmonic} \ldots) \\ u(x,y) = f(x,y) & \text{on } \partial\Omega \end{cases}$$

<u>Idea</u>: let's switch to polar coordinates $\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$

$u_{xx} + u_{yy} = \Delta u = 0$ becomes

$$\boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} = 0}$$

true for (r,ϑ) in Ω, i.e. $0 \leq r < 1 \qquad 0 \leq \vartheta < 2\pi$

We have $u(x,y) = u(r\cos\vartheta, r\sin\vartheta) = u(r,\vartheta)$ $\left(\begin{matrix}\text{abuse of}\\ \text{notation}\end{matrix}\right)$

$f(x,y) = f(\vartheta)$ 2π-periodic

known on $\partial\Omega$ ↳ only function of ϑ because on the boundary $r = 1$ (const.)

B.C. $u(1,\vartheta) = f(\vartheta)$ known 2π-periodic function

Look for $v(r,\vartheta) = R(r)\Theta(\vartheta)$ special case with separate (polar) variables (Analogously at page 64)

Plug this into $\Delta v = 0$ in polar coordinates, and obtain

$$1/r^2 R\Theta'' + \frac{1}{r}R'\Theta + R''\Theta = 0 \qquad \text{multiply all by } \frac{r^2}{R\Theta}$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$$\Theta'' + \lambda\Theta = 0$$

$$t^2 + \lambda = 0 \quad t^2 = -\lambda \quad t = \pm i\sqrt{\lambda} \;\dots \text{ like before}$$

$$\lambda = n^2 \quad \left(\begin{array}{c}\text{function at } \partial\Omega \\ \text{is } 2\pi\text{-periodic}\end{array}\right)$$

$\Theta(\vartheta) = \frac{1}{2}a_0$ if $n = 0$

$\Theta(\vartheta) = a_n \cos n\vartheta + b_n \sin n\vartheta$ if $n = 1,2,3,\dots$

$r^2R'' + rR' - n^2R = 0$ linear ODE 2nd order, non-constant coeff's homogeneous (BUT there is a substitution that changes it into linear ODE with constant coeff's, Euler's ODE)

The sol's are:

if $n = 0$

$$R(r) = c_1 + \cancel{c_2 \log r} \qquad (*)$$

if $n > 0$ $R(r) = c_1 r^n + \cancel{c_2 r^{-n}}$ $(**)$

We need to keep in mind that our separable variable solution $v(r,\vartheta) = R(r)\Theta(\vartheta)$ must be harmonic inside Ω, in particular harmonic functions are $C^\infty(\Omega)$. Now if $c_2 \neq 0$ in $(*)$ and/or $(**)$ we would have a discontinuity of $v(r,\vartheta)$ for $r = 0 \Rightarrow c_2 = 0$

THUS

$v(r,\vartheta) = r^n(a_n \cos n\vartheta + b_n \sin n\vartheta)$ $\left\{\begin{array}{l}\text{here we have embedded} \\ \text{the nonzero constant } c_1 \\ \text{into } a_n \text{ \& } b_n\end{array}\right\}$

Fourier: suppose $f(\vartheta) \overset{F.S.}{=} \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\vartheta + b_n \sin n\vartheta)$

$$\Rightarrow u(r,\vartheta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\vartheta + b_n \sin n\vartheta) \quad (\star)$$

This looks very simple (if we know how to write the F.S. for $f(\vartheta)$ then $u(r,\vartheta)$ is the same series with an extra factor r^n) but there are some possible computational difficulties.
For example, given $f(x,y)$ it might be not immediate to compute F.S. of $f(\vartheta)$; also, we need $u(x,y)$ in Cartesian coordinates; we need to convert back from the polar formula $(\star)$.

ex.1) $\Delta u = 0$ in $\Omega \equiv \{(x,y)\in\mathbb{R}^2\ x^2+y^2<1\}$
[P.70]

$$u(x,y) = x^2+y \quad \text{on } \partial\Omega \equiv \{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$$

solution $\underset{\text{on } \partial\Omega}{u(x,y)} = u(\underset{1}{\underset{\parallel}{r}},\vartheta) = f(\vartheta) = \cos^2\vartheta + \sin\vartheta =$

$$= \frac{1+\cos 2\vartheta}{2} + \sin\vartheta =$$

$$= \frac{1}{2} + \sin\vartheta + \frac{1}{2}\cos 2\vartheta \quad \text{F.S. with only 3 terms}$$

$$a_0 = 1 \quad b_1 = 1 \quad a_2 = \frac{1}{2}$$

all other $a_k = 0 \quad b_k = 0$

N.B. we could also compute

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\vartheta)\cos k\vartheta \, d\vartheta \quad \dots \quad \text{(less efficient)}$$

$$u(r,\vartheta) = \frac{1}{2} + r\sin\vartheta + \frac{r^2}{2}\cos 2\vartheta$$

General scheme $f(\vartheta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\vartheta + b_n \sin n\vartheta)$

$$u(r,\vartheta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\vartheta + b_n \sin n\vartheta)$$

$$u(r,\vartheta) = u(x,y) = \frac{1}{2} + y + \frac{1}{2}(x^2-y^2) \qquad \left(\begin{array}{l}\text{Remember} \\ \cos 2\vartheta = \cos^2\vartheta - \sin^2\vartheta\end{array}\right)$$

Next time we'll see more generally how to convert powers into frequencies.

ex. 2) [P.71] $\Delta u = 0$ in $\Omega \equiv \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$

$u(x,y) = \log(2+x)$ on $\partial\Omega \equiv \{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$

1) let's apply our solution Formula

$$f(x,y) = \log(2+x) = f(\vartheta) = \log(2+\cos\vartheta) \quad \text{EVEN FUNCTION} \quad \underline{\underline{\text{F.S.}}}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{where } a_n \overset{(*)}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} \log(2+\cos\vartheta)\cos n\vartheta$$

2) Instead of (*) we could try:

$$\log(2+\cos\vartheta) = \log\left(2\left(1+\frac{\cos\vartheta}{2}\right)\right) = \log 2 + \log\left(1+\frac{\cos\vartheta}{2}\right) =$$

$$= \log 2 + \frac{\cos\vartheta}{2} - \frac{\left(\frac{\cos\vartheta}{2}\right)^2}{2} + \frac{\left(\frac{\cos\vartheta}{2}\right)^3}{3} - \frac{\left(\frac{\cos\vartheta}{2}\right)^4}{4} + \dots$$

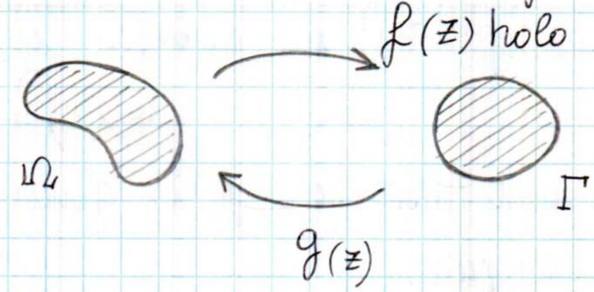
$\Rightarrow$ this approach might be more efficient to obtain the F.S. we need, provided we know how to convert powers of $\cos\vartheta$ to linear combinations of $\cos n\vartheta$ ...

$$\text{e.g. } \cos^2\vartheta = \frac{1+\cos 2\vartheta}{2}$$

$$\cos^3\vartheta = \dots$$

3) Third Possibility the formula we have obtained in polar coordinates leads also to a convolution formula (analogy with $\overline{\text{\\\\\\\\}}$) [NEXT PAGE]

4) Fourth possibility: use Riemann - Mapping theorem [P.73]

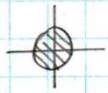


Review of last class:

LESS. 14 22/01/09

$u \equiv u(x,y)$ s.t. $u_{xx} + u_{yy} = 0$ in Ω

$\Omega \equiv \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$



(this "alone" defines harmonic functions $\Delta u = 0$)

B.C. $u(x,y) = f(x,y)$ on $\partial\Omega$

$\partial\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ CIRCLE

equivalently $u(\cos\vartheta, \sin\vartheta) = f(\vartheta)$ abuse of notation

$u(\rho,\vartheta)\big|_{\rho=1} = f(\vartheta)$

$x = \rho\cos\vartheta$

$y = \rho\sin\vartheta$

given (x,y) we can find (ρ,ϑ) and viceversa (actually ϑ is computed up to 2π-periodicity, e.g., $f(\vartheta + 2k\pi) = f(\vartheta)$ $k \in \mathbb{Z}$)

Using separation of variables and F.S. we have seen that if we can compute the Fourier coeff's of the known function $f(\vartheta)$ at $\partial\Omega$ i.e. $f(\vartheta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\vartheta + b_n \sin n\vartheta)$

then the solution u, in polar coordinates, is

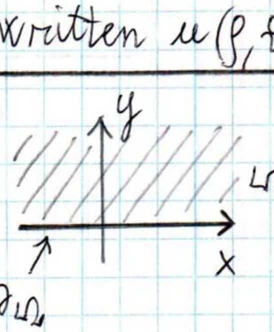
$$u(\rho,\vartheta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\vartheta + b_n \sin n\vartheta)$$

harmonic $u(x,y)$ in Ω written $u(\rho,\vartheta)$

A while ago, using F.T., we solved $u_{xx} + u_{yy} = 0$

[P. 55]

$u(x,0) = f(x)$



and we discovered that $u(x,y)$ is given by a convolution integral.

Theorem For any $\Omega \subseteq \mathbb{R}^2$, open and simply connected, then the solution of the problem $\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } \Omega \\ u(x,y) = \underset{\text{KNOWN}}{f} & \text{on } \partial\Omega \end{cases}$

Analogy with ODE, see P. 31R

is given by $\boxed{u(x,y) = P_\Omega * f}$

(a convolution integral, where the term P_Ω is called <u>Poisson's Kernel</u> relative to the domain Ω: each domain has its own Poisson Kernel).

Let us compute the Poisson Kernel for the <u>disc</u> $\Omega \equiv \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$

i.e. $\Omega \equiv \{(r,\vartheta) \in \mathbb{R}^2 : 0 \leq r < 1 , \vartheta \in [0, 2\pi)\}$

Let's start from the formula we obtained via F.S. which is

$$u(r,\vartheta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\vartheta + b_n \sin n\vartheta) =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{n=1}^{\infty} r^n \Big(\cos n\vartheta \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt + \sin n\vartheta \cdot$$

$$\cdot \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt\Big) =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\Big[1 + \sum_{n=1}^{\infty} 2r^n \underbrace{(\cos n\vartheta \cos nt + \sin n\vartheta \sin nt)}_{\substack{\cos(n(\vartheta - t)) \\ \cos n\alpha}}\Big]dt = (*)$$

$$[\alpha \stackrel{def}{=} \vartheta - t]$$

Let's simplify $[\ldots] = 1 + 2\sum_{n=1}^{\infty} r^n \cos n\alpha$

$$= 1 + 2\sum_{n=1}^{\infty} \mathrm{Re}\,(r^n e^{in\alpha})$$

(Euler's formula)

$$\boxed{\underline{\text{Remark}}\text{ (later)}: z + \bar{z} = 2\,\mathrm{Re}(z)}$$

$$= 1 + 2\,\mathrm{Re}\Big(\sum_{n=1}^{\infty} (\underbrace{re^{i\alpha}}_{<1})^n\Big) \underset{G.S.}{=\!=\!=} 1 + 2\,\mathrm{Re}\Big(\frac{1}{1-re^{i\alpha}} - 1\Big) =$$

<1 because of our disc

because we started the sum from $n=1$

$$= 1 + 2\,\mathrm{Re}\Big(\frac{\cancel{1}-\cancel{1}+re^{i\alpha}}{1-re^{i\alpha}}\Big) = 1 + \frac{re^{i\alpha}}{1-re^{i\alpha}} + \frac{re^{-i\alpha}}{1-re^{-i\alpha}}$$

$\Big\{$ to conjugate any formula we conjugate <u>all</u> the complex members in it and $\overline{\frac{re^{i\alpha}}{1-re^{i\alpha}}} = \frac{re^{-i\alpha}}{1-re^{-i\alpha}}$ $\Big\}$

$$= \frac{1}{(1-re^{i\alpha})(1-re^{-i\alpha})}\left\{(1-re^{i\alpha})(1-re^{-i\alpha}) + re^{i\alpha}(1-re^{-i\alpha}) + re^{-i\alpha}(1-re^{i\alpha})\right\}$$

$$= \frac{1}{(\dots)(\dots)}\left\{1 - r\underbrace{(\overset{z}{\overbrace{e^{i\alpha}}} + \overset{\bar{z}}{\overbrace{e^{-i\alpha}}})}_{2\mathrm{Re}(e^{i\alpha}) = 2\cos\alpha} + r^2 + re^{i\alpha} - r^2 + re^{-i\alpha} - r^2\right\} =$$

$$= \frac{1}{(\dots)(\dots)}\left\{1 - \cancel{2r\cos\alpha} + r^2 + \cancel{2r\cos\alpha} - 2r^2\right\} = \frac{1-r^2}{1-2r\cos\underset{(\theta - t)}{\underset{\parallel}{\alpha}} + r^2}$$

Now we go back to (*)

$$u(r,\vartheta) = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{f(t)}{1-2r\cos(\theta-t)+r^2}\,dt = P_r(\theta) * f$$

Poisson kernel for the unit disc Ω

Where

$$\boxed{P_r(\theta) = \frac{\frac{1}{2\pi}(1-r^2)}{1-2r\cos\theta + r^2}}$$

N.B. If Ω is a disc of radius R

$$\Omega \equiv \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < R^2\}$$

we have (via a change of variable):

$$u(\rho,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n\cos n\theta + b_n \sin n\theta) =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-t)}\,dt$$

Ex. 1)
$$\begin{cases} u_{xx}+u_{yy} = 0 & \text{in } \Omega \equiv \{(x,y) : x^2+y^2 < R^2\} \\ u(x,y) = x+y & \text{on } \partial\Omega \equiv \{(x,y) : x^2+y^2 = R^2\} \end{cases}$$

$$f(x,y) = x+y = R\cos\theta + R\sin\theta \qquad \theta \in [0,2\pi)$$

The F.S. of f, in this particular example, is very easy: all the a_n and b_n are 0 except $a_1 = R$, $b_1 = R$

by our first (F.S.) solution

$$u(r,\theta) = \left(\frac{r}{R}\right)^1 (R\cos\theta + R\sin\theta) = r\cos\theta + r\sin\theta = x + y$$

$\Rightarrow$ In this case $f(x,y)$, which is the data at the boundary, <u>coincides</u> with our solution $u(x,y)$.

<u>This was predictable</u> because $x+y$ is itself harmonic $\Delta(x+y) = 0$ on $\mathbb{R}^2$. [CFR. 56]

<u>EX. 2)</u>
$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } \Omega \equiv \{(x,y) : x^2 + y^2 < R^2\} \\ u(x,y) = x^2 + y & \text{on } \partial\Omega \end{cases}$$

<u>N.B.</u> unlike the f in ex.1 this $f = x^2 + y$ IS NOT harmonic ($f_{xx} = 2$; $f_{yy} = 0$ $\Delta f = 2 \neq 0$)

$$f(\theta) = R^2\cos^2\theta + R\sin\theta = R^2 \frac{1+\cos 2\theta}{2} + R\sin\theta$$

<u>Remark</u> $\frac{1+\cos 2\theta}{2} = \cos^2\theta$ special case of formulas that today we'll discuss

$$= \frac{R^2}{2} + R\sin\theta + \frac{R^2}{2}\cos 2\theta \quad \left(\begin{array}{l}\text{F.S.} \\ \text{finite sum}\end{array}\right)$$

(only a_0, a_2, b_1 are $\neq 0$, <u>all</u> others $= 0$)

$$u(r,\theta) = \frac{R^2}{2} + \left(\frac{r}{R}\right)^1 (R\sin\theta) + \left(\frac{r}{R}\right)^2 \frac{R^2}{2}\cos 2\theta =$$

$$= \frac{R^2}{2} + r\sin\theta + \frac{r^2}{2}\cos 2\theta = \quad \{\cos 2\theta = \cos^2\theta - \sin^2\theta\}$$

$$u(x,y) = \frac{R^2}{2} + y + \frac{1}{2}(x^2 - y^2)$$ (special case of formulas we'll discuss)

Now, as it must be, this $u(x,y)$ is harmonic

<u>Check</u> $u_x = \frac{1}{2} 2x = x \quad u_{xx} = 1 \qquad \Delta u = 0$ OK!

$u_y = 1 - \frac{1}{2} 2y = 1 - y \quad u_{yy} = -1$

If we restrict this harmonic polynomial $u(x,y)$ (in $\mathbb{R}^2$) to $\partial\Omega$

$$= \begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases} \quad \text{get} \quad \frac{R^2}{2} + R\sin\theta + \frac{1}{2}R^2(\cos^2\theta - \sin^2\theta) =$$

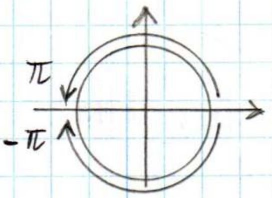
$$= \frac{R^2}{2}(1+\cos 2\theta) + R\sin\theta = x^2 + y \quad \checkmark$$

In other words, the equality

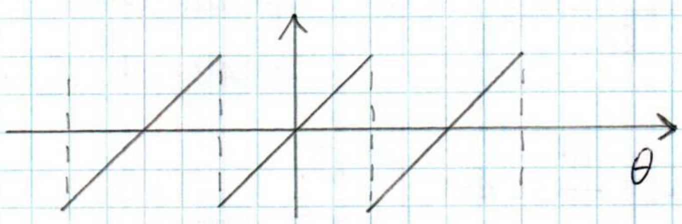
$$\frac{R^2}{2} + y + \frac{1}{2}(x^2 - y^2) = x^2 + y \quad \text{is false in general}$$

BUT true if $x = R\cos\theta$ $y = R\sin\theta$

<u>Ex.3)</u> $\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } \Omega \equiv \{(x,y) : x^2+y^2 < 1\} \\ u(x,y) = f(\theta) = \theta & \text{for } \theta \in (-\pi, \pi) \end{cases}$



N.B. at $\theta = \pi$ the B.C. $f(\theta)$ has a jump



$f(\theta)$ seen as a 2π-periodic function of θ is a SAWTOOTH function

We have seen that

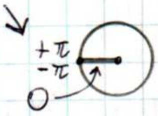
$$f(\theta) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta$$

$$f(\theta) = 2\left\{\sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \frac{1}{4}\sin 4\theta + \dots\right\}$$

$$u(r,\theta) = \sum_{n=1}^{\infty} \left(\frac{r}{1}\right)^n \frac{2}{n}(-1)^{n+1} \sin n\theta = 2\sum_{n=1}^{\infty} \frac{r^n}{n}(-1)^{n+1} \sin n\theta$$

<u>Obs.1</u>: in example 1 and 2 our solution $u(x,y) = u(r,\theta)$ extended outside Ω to all of $\mathbb{R}^2$, in this example 3 it <u>does NOT</u>.

Obs. 2: $u(r,\pi)=0 \quad \forall r \in \Omega \quad 0<r<1$ 0 is the midpoint of the jumps of $f(\theta)$



Obs. 3: If we are asked to write our solution in Cartesian coordinates we need a recipe to convert $r^n \sin n\theta$ and $r^n \cos n\theta$ into powers of x, y. If we do this we obtain the Taylor series in two variables for $u(x,y)$

$$(re^{i\theta})^n \overset{\text{Euler}}{=} r^n(\cos n\theta + i \sin n\theta)$$

$$x+iy = re^{i\theta} = r(\cos\theta + i\sin\theta) = \underbrace{r\cos\theta}_{x} + i\,\underbrace{r\sin\theta}_{y}$$

$$(x+iy)^n = r^n \cos n\theta + i r^n \sin n\theta \quad \text{[DE MOIVRE THEOREM]}$$

$$\Rightarrow \boxed{\begin{array}{l} r^n \cos n\theta = \mathrm{Re}\,(x+iy)^n \\ r^n \sin n\theta = \mathrm{Im}(x+iy)^n \end{array}}$$

where (P.C.) $x = r\cos\theta$, $y = r\sin\theta$

Using this idea let's write the first few terms of the Taylor series in two variables x, y for the solution $u(x,y)$ of ex.3: $u(x,y) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \mathrm{Im}(x+iy)^n =$

$$u(x,y) = 2\Big\{1\cdot y - \tfrac{1}{2}(2xy) + \tfrac{1}{3}(3x^2y - y^3) - \tfrac{1}{4}(4x^3y - 4xy^3) +$$
$$+ \tfrac{1}{5}(5x^4y - 10x^2y^3 + y^5) - \tfrac{1}{6}(6x^5y - 20x^3y^3 + 6xy^5) + \ldots$$

(this series converges only if $r<1$)

Ex. 4) $u_{xx} + u_{yy} = 0$ in $\Omega \equiv \{(x,y) : x^2+y^2<1\}$

$u(x,y) = \log(2+x)$ on $\partial\Omega \equiv \{(x,y) : x^2+y^2 = 1\}$

Probably in this case it's easier to write the solution as a convolution with the Poisson kernel instead of writing series (ex. 1, 2, 3).

We need the F.S. of $f(\theta)$ to start

$x = \cos\theta$ on $\partial\Omega$ $f(\theta) = \log(2 + \cos\theta)$ $\quad b_n = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \log(2+\cos\theta)\cos n\theta \, d\theta$$ (try $\int$ by parts, substitution, ...)

feasible, BUT the complexity of computation increases with n.

Other possibility $f(\theta) = \log(2+\cos\theta) =$

$$= \log\left(2\left(1+\frac{\cos\theta}{2}\right)\right) = \log 2 + \log\left(1 + \underbrace{\frac{\cos\theta}{2}}_{|\dots| \le \frac{1}{2}}\right) =$$

$$(*) \quad = \log 2 + \left(\frac{\cos\theta}{2}\right) - \frac{\left(\frac{\cos\theta}{2}\right)^2}{2} + \frac{\left(\frac{\cos\theta}{2}\right)^3}{3} - \frac{\left(\frac{\cos\theta}{2}\right)^4}{4} +$$

using $\log(1+t) = t - t^2/2 + t^3/3 - t^4/4 + \dots$ Radius of conv. $= 1$

$(*)$ is almost a F.S., BUT with powers instead of frequencies.

We'll see how to convert powers to frequencies, BUT, the series of series that we so obtain looks computationally hard to handle => use convolution instead!

$$\cos^2\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2 = \frac{1}{4}\left(e^{2i\theta} + e^{-2i\theta} + 2e^{i\theta - i\theta}\right)$$

$$= \frac{1}{4}(2\cos 2\theta + 2) = \frac{1+\cos 2\theta}{2}$$

$$\cos 3\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^3 = \frac{1}{8}\left(e^{3i\theta} + 3e^{\overbrace{2i\theta - i\theta}^{i\theta}} + 3e^{\overbrace{i\theta - 2i\theta}^{-i\theta}} + e^{-3i\theta}\right)$$

$$= \frac{1}{8}(2\cos 3\theta + 6\cos\theta) = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

YOU GET THIS FROM TH NEWTON BINOMIAL ...

In general $\boxed{\cos^n\theta} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} e^{i(n-k)\theta} e^{i(-k)\theta} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} e^{i(n-2k)}$

Remember $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ --- pair terms to obtain cosine!

In general harmonic functions $u = u(x,y)$

$$(x,y) \in \Omega \subseteq \mathbb{R}^2$$

[P54] satisfy the Maximum Principle (true even for $n>2$ variables)
Let us prove it for 2 variables.

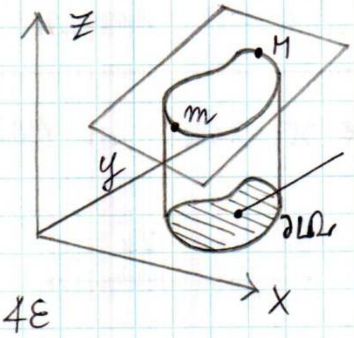
Theor. Ω bounded region (open, connected) $\subseteq \mathbb{R}^2$; let $\bar{\Omega} = \Omega \cup \partial\Omega$ (closure of Ω)

Suppose $u(x,y) \in C(\bar{\Omega})$ and $\Delta u = 0$ in Ω (harmonic)

then $\max\limits_{(x,y)\in\bar{\Omega}} u(x,y)$ and $\min\limits_{(x,y)\in\bar{\Omega}} u(x,y)$

(Which do exist because of Weierstrass 2 var's theorem)

are attained both on $\partial\Omega$.



Proof: suppose $\varepsilon > 0$ (small)

$$v(x,y) \stackrel{def}{=} u(x,y) + \varepsilon(x^2+y^2)$$

$$v_{xx} + v_{yy} = \underbrace{\cancel{u_{xx} + u_{yy}}}_{\text{because it is harmonic} \to 0} + \varepsilon(2+2) = 4\varepsilon$$

Suppose $(x_0, y_0) \in \bar{\Omega}$ is a point (not necessarily unique)
Where the (unique) $\max\limits_{(x,y)\in\bar{\Omega}} v(x,y)$ is attained

$$\max_{(x,y)\in\bar{\Omega}} v(x,y) = v(x_0,y_0)$$

if $(x_0,y_0) \in \Omega$, the absolute maximum of v is also a relative maximum $\Rightarrow$ it must satisfy

$$v_{xx}(x_0,y_0) \leq 0 \quad \text{and} \quad v_{yy}(x_0,y_0) \leq 0$$

These inequalities contradict $v_{xx} + v_{yy} = 4\varepsilon$ for $(x,y) \in \Omega$

$\Rightarrow (x_0, y_0) \in \partial\Omega$

Set $M = \max\limits_{(x,y)\in\partial\Omega} u(x,y)$ then $\max\limits_{(x,y)\in\bar{\Omega}} u(x,y) \leq \max\limits_{(x,y)\in\bar{\Omega}} v(x,y) \leq$

$$\leq M + \varepsilon \max_{(x,y)\in\partial\Omega} (\tilde{x}^2+\tilde{y}^2)$$

$\max_{(x,y)\in\bar{\Omega}} u(x,y) \leq M \Rightarrow$ the max of $u(x,y)$ is attained on the boundary

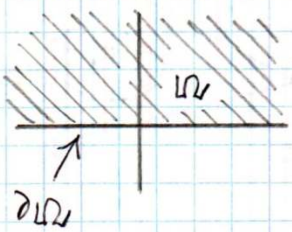
If we apply this same reasoning to $-u(x,y)$ (which is also harmonic), we obtain the same result about the $\min_{(x,y)\in\bar{\Omega}} u(x,y)$

The Maximum Principle is important:

(1) because it's a characteristic property of all harmonic functions;

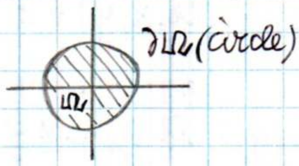
(2) using it we can prove that the B.V. problems for $\Delta u = 0$ in Ω are well-posed.

We have studied in reasonable detail two cases:



$\Delta u = 0$ on Ω

$u = f$ known on $\partial\Omega$



$\Delta u = 0$ in Ω

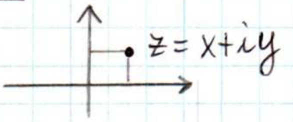
$u = f$ known on $\partial\Omega$

We could study other cases on a case by case fashion.

To study all of these <u>elliptic</u> PDE boundary value problems for all <u>simply connected</u> <u>domains</u> Ω there is a unified method in the case of <u>2 variables</u> via the "<u>Riemann Mapping</u>" <u>theorem</u>.

If the number of variables is $n > 2$ OR if the domain Ω is not simply connected the problem is studied, but it's more difficult.

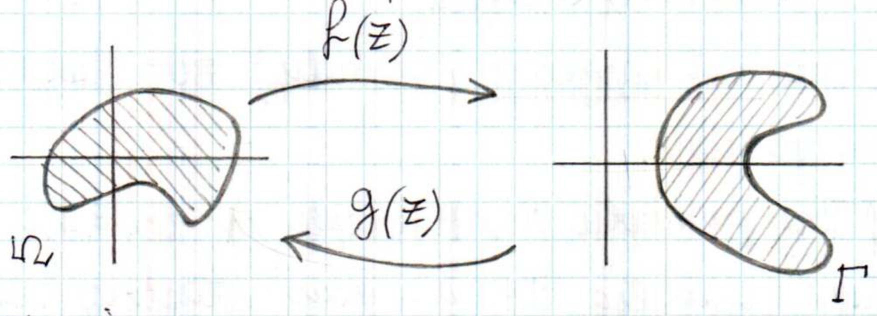
Remark If $f(z)$ is holomorphic in $\Omega \underset{\subseteq}{\subseteq} \mathbb{C} \subseteq \mathbb{R}^2$



if we let the complex variable $z = x+iy$ vary inside Ω, the $w = f(z)$ moves inside another open simply connected set Γ. We write $f(\Omega) = \Gamma$.

If the inverse function $g(z)$ is well defined on Γ $\left(g(f(z)) = z = f(g(z))\right)$ then g is also holomorphic

and we have a double map between sets of the plane $\mathbb{R}^2 \equiv \mathbb{C}$



Conformal Mapping

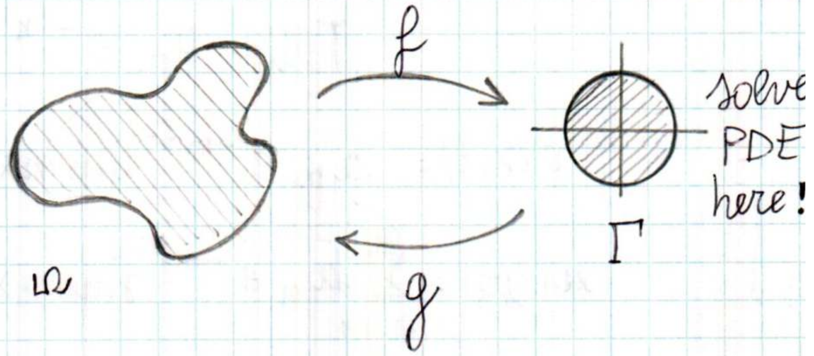
Riemann Mapping Theorem (No Proof)

If $\Omega \subseteq \mathbb{C} \equiv \mathbb{R}^2$ is simply connected open $(\neq \emptyset)$ then $\exists$ a conformal mapping ($f(z)$ and $g(z)$ inverse of each other and both holomorphic) such that $f(\Omega) = \Gamma$ and $g(\Gamma) = \Omega$ for any other simply connected region Γ.

Theorem (easy) $f(u(x,y))$ with u harmonic, f holomorphic

is also harmonic.

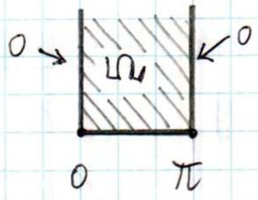
Method of solution:



Parabolic PDE

[CFR. P. 51R]

$$u = u(x,t)$$



$$u_t = u_{xx}$$

$$\text{in } \Omega \equiv \left\{ (x,y) : \begin{array}{l} x \in (0;\pi) \\ t > 0 \end{array} \right\}$$

$$\underset{\text{KNOWN}}{u(x,0) = u_0(x)} \quad \text{if } x \in (0;\pi)$$

$$u(0,t) = u(\pi,t) = 0 \quad \text{for } t > 0$$

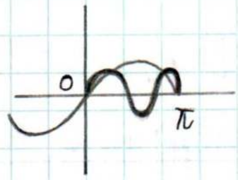
$$u(x,t) \in C(\bar{\Omega})$$

These u are <u>not harmonic</u> (usually) BUT they do satisfy a maximum principle.

<u>Proof</u> of the parabolic Max Principle is th. 8.4.2 p. 151. If you choose parabolic PDE as your starting topic, study this proof. If you do not choose this, I won't ask this proof.

$v(x,t) = X(x)\,T(t)$ special case with separable variable

$$X(x)\,T'(t) = X''(x)\,T(t) \qquad X(0) = X(\pi) = 0$$



$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -\lambda$$

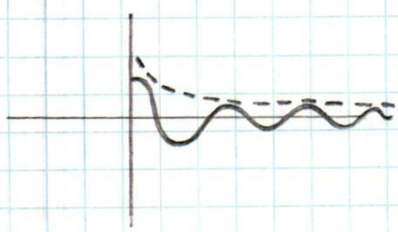
$X'' + \lambda X = 0$ ODE's associated with $T' + \lambda T = 0$

$$\lambda = n^2 \quad X(x) = \sin nx$$

$$T(t) = b_n e^{-n^2 t}$$

$$v(x,t) = b_n e^{-n^2 t} \sin nx$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$$



$$u(x,0) = u_0(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad b_n = \int \ldots$$

To compute the Fourier coeff's $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\overset{\text{ODD}}{u_0(x)} \overset{\text{ODD}}{\sin nx}}_{\text{EVEN}} \, dx$
do the odd continuation of $u_0(x)$
to be 2π-periodic or $\qquad b_n = \frac{2}{\pi} \int_0^{\pi} u_0(x) \sin nx \, dx$

Can be written also as convolution (Heat or Gauss Kernel in)

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