The last time we defined the following notations:

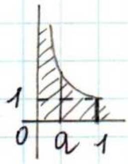
$f \in L^p(A) \iff \|f\|_p$ is <u>finite</u>

$1 \leq p \leq \infty$ $\quad$ $\boxed{\|f\|_p = \left(\int_A |f(x)|^p \, dx \right)^{1/p}}$

often $p=1$
$p=2$

<u>ex.1)</u> $f(x) = \frac{1}{\sqrt[3]{x}} \in L^1((0;1))$ $\quad A \equiv (0;1)$

<u>check</u> $\|f\|_1 = \int_0^1 \left| \frac{1}{x^{1/3}} \right| dx = \int_0^1 x^{-1/3} \, dx$



this is a generalized integral

N.B. $\lim_{x \to 0^+} \frac{1}{\sqrt[3]{x}} = +\infty$

$$\int x^{-1/3} \, dx = \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c = \frac{3}{2} x^{2/3} + c$$

Now $\int_0^1 \frac{1}{\sqrt[3]{x}} \, dx \overset{def.}{=} \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt[3]{x}} \, dx = \lim_{a \to 0^+} \left(\frac{3}{2} 1^{2/3} - \frac{3}{2} a^{2/3} \right) =$

$= 3/2$ FINITE VALUE

i.e. on $A \equiv (0;1)$ $\quad \left\| \frac{1}{\sqrt[3]{x}} \right\|_1 = 3/2$

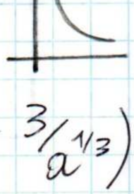
A similar computation shows that $\left\| \frac{1}{\sqrt[3]{x}} \right\|_{L^2(0;1)}$ is <u>finite</u>

$\left\| \frac{1}{\sqrt[3]{x}} \right\|_2 = \left(\int_0^1 x^{-2/3} \, dx \right)^{1/2} = \ldots$ FINITE VALUE

Let's show that the same $f(x) = \frac{1}{\sqrt[3]{x}} \notin L^4(0;1)$

$\|f\|_4 = \left(\int_0^1 \left(x^{-1/3} \right)^4 dx \right)^{1/4} = \left(\int_0^1 x^{-4/3} \, dx \right)^{1/4}$

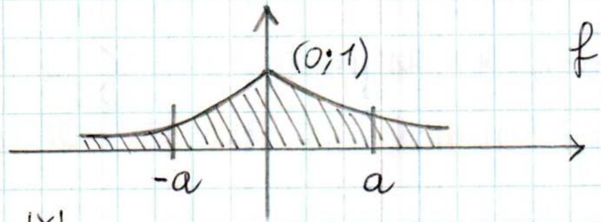
Now $\int x^{-4/3} \, dx = \frac{x^{-4/3+1}}{-4/3+1} = -3 \, x^{-1/3} = -3/x^{1/3}$



$\int_0^1 x^{-4/3} \, dx \overset{def.}{=} \lim_{a \to 0^+} \int_a^1 x^{-4/3} \, dx = \lim_{a \to 0^+} \left(-3/1^{1/3} + 3/a^{1/3} \right) = +\infty$

<u>Moral</u> (from this example): the same f may belong to $L^p(A)$ or not belong depending on p.

ex. $f(x) = e^{-|x|}$



$f(x) = f(-x)$ EVEN FUNCTION

$f \in L^1(\mathbb{R}) \iff \int_{-\infty}^{+\infty} e^{-|x|} dx$ FINITE

'cause the function is even

$$\int_{-\infty}^{+\infty} e^{-|x|} dx \overset{def.}{=} \lim_{a \to +\infty} \int_{-a}^{a} e^{-|x|} dx = 2 \lim_{a \to +\infty} \int_0^a e^{-x} dx =$$

$$= 2 \lim_{a \to +\infty} \left[-e^{-x}\right]_0^a = 2 \lim_{a \to +\infty} (\underbrace{-e^{-a}}_{\to 0} + e^0) = 2$$

Actually, $e^{-|x|} \in L^p(\mathbb{R})$ for all $1 \leq p \leq \infty$ (included the infinitive). Proof:

$$\|e^{-|x|}\|_p = \left(2 \lim_{a \to +\infty} \int_0^a e^{-px} dx\right)^{1/p} = \left(2 \lim_{a \to +\infty} \left[-\frac{1}{p} e^{-px}\right]_0^a\right)^{1/p} =$$

$$= \left(\frac{2}{p}\right)^{1/p}$$ FINITE

$$\lim_{p \to +\infty} \left(2/p\right)^{1/p} = 1$$ this is the maximum height of the function (basically it's the supremum of the abs.val. of the function).

$$= \|e^{-|x|}\|_{L^\infty(\mathbb{R})}$$

ESSENTIAL SUPREMUM
(see page 10R)

ex. $f(x) = 1/x$ $\quad A \equiv [1; +\infty)$ $\quad f \notin L^1(A)$

$f \in L^p(A)$ if $p > 1$

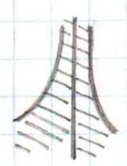
Proof: $\|f\|_p = \left(\int_1^{+\infty} \left(\frac{1}{x}\right)^p dx\right)^{1/p}$ if $p = 1$ we get

$$\int_1^{+\infty} 1/x \, dx \overset{def.}{=} \lim_{b \to \infty} \int_1^b 1/x \, dx = \lim_{b \to +\infty} (\log b - \log 1) = +\infty$$

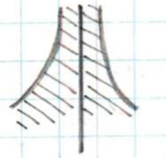
$\Rightarrow f \notin L^1(A)$

If $p > 1$ $\quad \int_1^{+\infty} 1/x^p \, dx \overset{def.}{=} \lim_{b \to +\infty} \int_1^b x^{-p} dx = \ldots$ FINITE

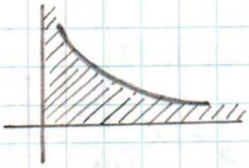
Remark if $f(x)$ blows up near a vertical asyntote $x = x_0$, and is asyntotic to $\frac{1}{|x - x_0|^\alpha}$, its integral in a neighborhood of x_0 is FINITE if $\alpha < 1$.



Also if $f(x)$ is asymptotic to $1/|x|^\alpha$ at $\pm\infty$, then its integral in a neighborhood of $+\infty$ or $-\infty$ is FINITE if $\alpha > 1$.



FINITE AREA FOR $\alpha < 1$



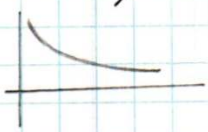
FINITE AREA FOR $\alpha > 1$

> in particular $f(x) = 1/x$
> is not in L^1 near 0
> " " " " " ∞
> $\int_0^1 1/x \, dx = +\infty$
> $\int_1^{+\infty} 1/x \, dx = +\infty$

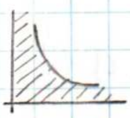
ex. $f(x) = 1/x^2$ then $f \in L^1$ near ∞; $f \notin L^1$ near 0

$$\int_1^{+\infty} 1/x^2 \, dx \text{ FINITE}$$
$$\int_0^1 1/x^2 \, dx = +\infty$$

N.B. These rules of integrability with $\alpha > 1$ at ∞ and $\alpha < 1$ at a finite value $x = x_0$ are useful BUT are not all powerful (do not solve all cases !!)

For example $f(x) = \dfrac{1}{x(\log x)^2}$  goes to 0 as $x \to +\infty$ faster than $1/x$, but slower than any $1/x^\alpha$ for fixed $\alpha > 1$.

ex. $\dfrac{1}{\sqrt{x}(1+x)} = f(x)$ I claim that $\int_0^{+\infty} f(x)\,dx$ is FINITE, i.e., $f \in L^1((0;+\infty))$



Proof: $\displaystyle\int \frac{1}{\sqrt{x}(1+x)} dx = \int \frac{2t\,dt}{t(1+t^2)} = 2 \operatorname{Arctg} t = 2 \operatorname{Arctg} \sqrt{x}$

$(\sqrt{x} = t,\ x = t^2,\ dx = 2t\,dt,\ x = 0\ t = 0,\ x \to +\infty\ t \to +\infty)$

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(1+x)} dx = \lim_{\substack{b \to \infty \\ a \to 0^+}} \int_a^b f(x)\,dx = 2(\underbrace{\operatorname{Arctg} b}_{\to \pi/2} - \overbrace{\operatorname{Arctg} a}^{\to 0})$$

$= \pi$ so $\|f\|_{L^1(0;+\infty)} = \pi$