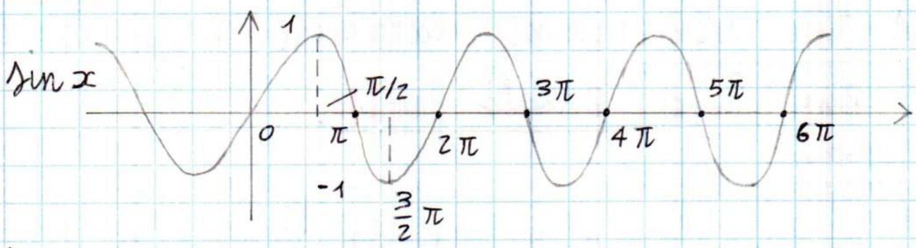
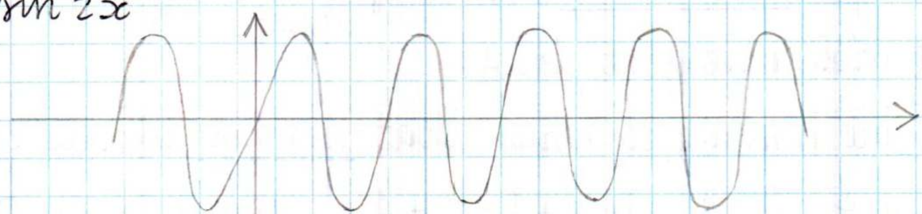
Last time we said a few things about power series. They're series of functions $\sum_{k=0}^{\infty} f_k(x)$ where $f_k(x) = c_k(x-x_0)^k$; in particular we observed that we have a Radius of convergence R and if $R>0$ the sum of these series are C^∞ functions $f(x)$ (actually holomorphic functions) and $c_k = \frac{f^{(k)}(x_0)}{k!}$.
We'll say more later about power series.
Another very important kind of series of functions are
FOURIER SERIES (F.S.)

F.S. $\boxed{\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)}$

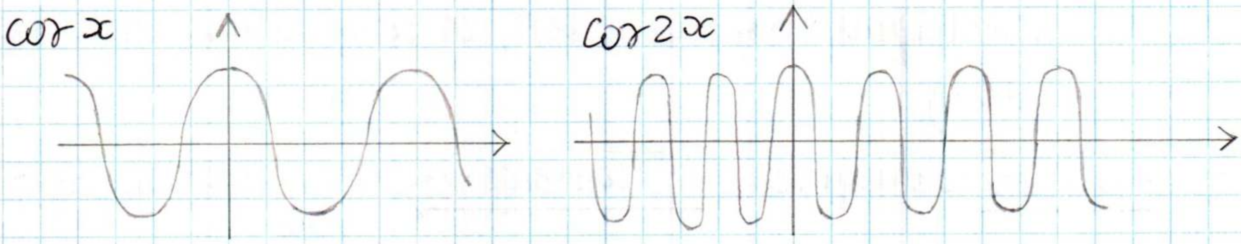
Remark: The sinusoidal functions look like:



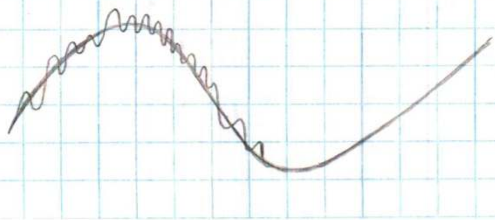


(oscillates twice as quickly)

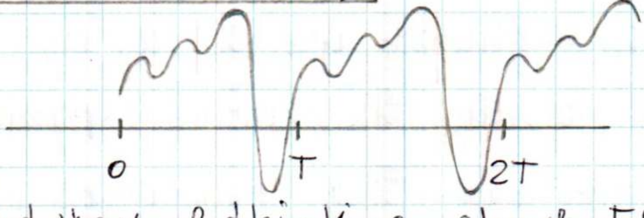
Fourier analysis is also called Harmonic Analysis



F.S. is a superposition of "sinus waves" of different amplitudes and different frequencies (integer multiples of a fundamental frequency).

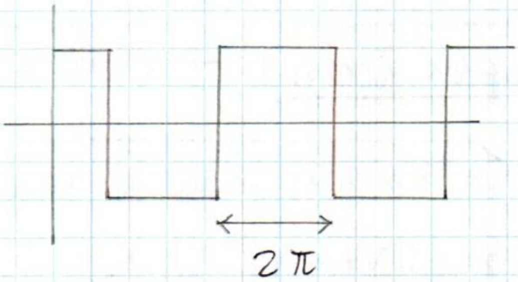


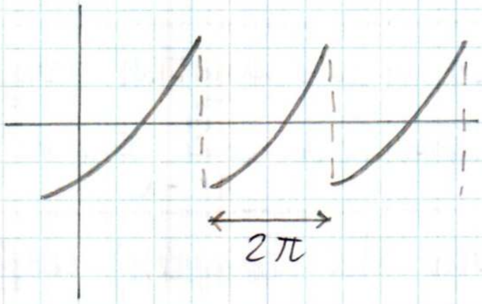
Def. A function $f(x)$ defined for $\forall x \in \mathbb{R}$ is called T-periodic if $\boxed{f(x) = f(x+nT)}$ $\forall n \in \mathbb{Z}$

graphically 

[Sound is a very good way of thinking about F.S.]

We will see that this kind of series can represent (with its sum) any "reasonable" 2π-periodic function $f(x)$ (such that $f(x) = f(x+2\pi n)$). Examples:





We'll learn how to compute the Fourier coefficients a_n and b_n (amplitudes) starting from $f(x)$.

In applications a F.S. has an extra parameter $T>0$ (period of our function) and becomes $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{2\pi}{T} nx + b_n \sin \frac{2\pi}{T} n$

In our mathematical presentation we'll often choose $T = 2\pi$ for simplicity.

NB.: In any case remember that a T-periodic function is a 2π-periodic function re-scaled horizontally

$$f(x) \rightleftarrows f\left(\frac{2\pi}{T} x\right)$$

Def. A trigonometric polynomial of degree N is the expression

$$\frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k \cos kx + b_k \sin kx \right)$$

Its coefficients are $a_0, a_1, a_2, \dots, a_N$ and $b_1, b_2, \dots, b_N$

More precisely the previous expression is the real-form of a 2π-periodic trig. polynomial of degree N.

Note that the N-th partial sums of a F.S. are exactly such an expression.

For trig. polynomials of period T we have:

$$\frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k \cos \frac{2\pi}{T} kx + b_k \sin \frac{2\pi}{T} kx \right)$$

There is also another way (complex form) of writing trig. polynomials: $\sum_{k=-N}^{+N} c_k e^{ikx}$ This is a bilateral sum

Given the (complex) coefficients (2N+1 of them) $c_{-N}, c_{-N+1}, \dots, c_{-1}, c_0, \dots, c_N$ we can find the corresponding (real) coeff.:

$\underbrace{a_0, a_1, \dots, a_N}_{N+1}$ and $\underbrace{b_1, b_2, \dots, b_N}_{N}$

Let's study this connection:

$$\sum_{k=-N}^{+N} c_k e^{ikx} = c_{-N} e^{-iNx} + \dots + c_{-1} e^{-ix} + c_0 e^0 + c_1 e^{ix} + \dots + c_N e^{iNx} \quad (2N+1 \text{ terms}) =$$

$$= c_0 + \sum_{k=1}^{N} \left(c_k \underset{(\cos kx + i \sin kx)}{e^{ikx}} + c_{-k} \underset{(\cos kx - i \sin kx)}{e^{-ikx}} \right)$$

Now we use the Euler's formula

$$= c_0 + \sum_{k=1}^{\infty} \left[\underbrace{(c_k + c_{-k})}_{a_k} \cos kx + \underbrace{i(c_k - c_{-k})}_{b_k} \sin kx \right]$$

$$(1) \quad \boxed{\begin{cases} a_k = c_k + c_{-k} \\ b_k = i(c_k - c_{-k}) \end{cases}}$$

$$\begin{cases} a_k = c_k + c_{-k} \\ i b_k = -c_k + c_{-k} \end{cases} \Rightarrow \begin{cases} a_k + i b_k = 2 c_{-k} \\ a_k - i b_k = 2 c_k \end{cases}$$

(We multiplied all by i)

Summing & subtracting

$$② \left\{ \boxed{\begin{aligned} c_k &= \frac{a_k - i b_k}{2} \\ \underline{c}_k &= \frac{a_k + i b_k}{2} \end{aligned}} \right.$$

$k = 0, 1, 2, \dots, N$ are complex conjugate

If our trig. polynomial has real coeff. a_k and b_k then its complex version has $c_k = \frac{a_k - i b_k}{2}$ for $k = 1, 2, \dots, N$

and $\underline{c}_k = \overline{c_k} = \frac{a_k + i b_k}{2}$

$a_0 = 2c_0 \Rightarrow c_0 = \frac{a_0}{2}$ $\qquad b_0 = 0 \equiv i(c_0 - c_0)$ (got from the ①)

$S_N(x)$ partial sums of order N of F.S. is equal to

$= \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx) = \sum_{k=-N}^{+N} c_k e^{ikx}$ ⇐ bilateral sum

and from the a_k and b_k we can compute c_k and vice-versa via ① & ②.

Given a 2π-periodic function $f(x)$ its Fourier coefficients satisfy ① & ②

$$(*) \quad \boxed{c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx} \Longrightarrow \text{from } c_k \text{ can get } a_k \text{ \& } b_k$$

Otherwise

$$\boxed{a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx}$$

$$\boxed{b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx}$$

Let's prove $(*)$ in the special case when $f(x) = \sum_{n=-N}^{+N} c_n e^{in}$ is a trig. pol. of degree N.

Apply $(*)$ to this $f(x) = \sum_{n=-N}^{+N} c_n e^{inx}$, get

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\left(\sum_{n=-N}^{N} c_n e^{inx} \right)}_{f(x)} e^{-ikx} \, dx =$$

$$= \frac{1}{2\pi} \sum_{n=-N}^{+N} c_n \underbrace{\int_{-\pi}^{\pi} e^{i(n-k)x} \, dx}_{\text{let's analyze this term...}}$$

<u>Claim</u> $\displaystyle\int_{-\pi}^{\pi} e^{i(n-k)x}\,dx = \begin{cases} 0 & \text{if } n \neq k \\ 2\pi & \text{if } n = k \end{cases}$

then only one of the $2N+1$ terms in the sum is $\neq 0$ and its value is $\frac{1}{\cancel{2\pi}} c_k \cdot \cancel{2\pi} = c_k$ Q.E.D.

Proof of claim

We use the Euler's Formula

$$\int_{-\pi}^{\pi} e^{i(n-k)x}\,dx = \int_{-\pi}^{\pi}\left[\cos(n-k)x + i\sin(n-k)x\right]dx = 0$$

because of the cancellation in the periodic sin, cos, .. $(n-k) \neq 0$

Assume $n-k = m \neq 0$ integer non-zero:

$$\int_{-\pi}^{\pi} e^{imx}\,dx = \left[\frac{e^{imx}}{im}\right]_{-\pi}^{\pi} = \frac{1}{im}\left(e^{im\pi} - e^{im(-\pi)}\right)$$

N.B. $\begin{pmatrix} e^{i\pi} = -1 \\ e^{-i\pi} = -1 \end{pmatrix}$ $\quad = \frac{1}{im}\left((-1)^m - (-1)^m\right) = 0$

Instead, if $n = k$ $(m=0)$ we have the integral $\int_{-\pi}^{\pi} e^{i\cdot 0\cdot x}\,dx =$

$$= \int_{-\pi}^{\pi} 1\,dx = [x]_{-\pi}^{\pi} = 2\pi \quad \text{Q.E.D.}$$

<u>Fourier's idea</u>: Given <u>any</u> 2π-periodic function (not just polyn.) let's define $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx$

<u>if</u> $f \in L^1(-\pi;\pi)$, <u>then</u> also $f(x)\,e^{-inx} \in L^1(-\pi;\pi)$

(<u>because</u> $|f(x)\,e^{-inx}| = |f(x)|\,|e^{-inx}| = |f(x)|\,\underbrace{|\cos nx - i\sin nx|}_{= 1 = (\cos^2 nx + \sin^2 nx)^{1/2}}$

$= |f(x)|$

MODULUS OF A COMPLEX NUMBER!

The ∞ many coeff. c_k are well defined.

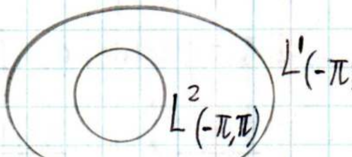
<u>Question</u>: if f is 2π-periodic, and $f \in L^1(-\pi;\pi)$ does its F.S. $\sum_{k=-\infty}^{+\infty} c_k e^{ikx}$ converge to $f(x)$?

<u>Answer</u>: it depends..

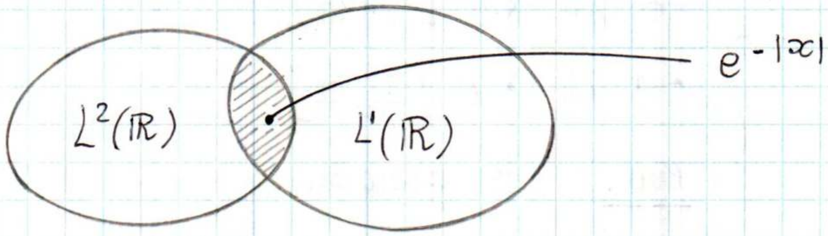
It always converges in the L^2-sense,

i.e. $\lim_{N\to\infty} \| S_N(x) - f(x) \|_2 = 0$

<u>Th.</u> if A is a <u>bounded</u> interval (like e.g. $(-\pi;\pi)$) then
$L^\infty(A) \subset L^p(A) \subset \dots \subset L^1(A)$ for $1 < p < \infty$



<u>NB:</u> this is FALSE if A is not bounded!
In particular (relevant for Fourier theory) $L^2(-\pi;\pi) \subsetneq L^1(-\pi$
<u>BUT</u> $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ neither space contains the other

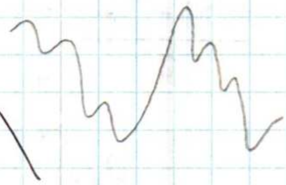


$\boxed{\text{So, for F.S., } \underline{\text{if}}\ f \in L^1(-\pi;\pi)\ \underline{\text{then}}\ f \in L^2(-\pi;\pi) \text{ and its F.S converges to } f \text{ in } L^2.}$ If f has some specific smoothnes properties OR if $c_n \to 0$ "quickly enough", then the F.S. of f converges to f also in other ways (pointwise, absolutely, uniformly ...). We'll see theorems of this kin [CFR P.

<u>N.B.</u> In a F.S. the "higher frequency" terms that we sum MUST have smaller & smaller amplitudes (size of c_n's) for the series to converge.

<u>Also</u> we have here one instance of the LOCAL-GLOBAL principle in Fourier Analysis (that we'll study more later): if a periodic f is very smooth, then the $c_n \to 0$ very quickly as $n \to \infty$ (and viceversa).

$$f(x) = \sum_{n=-\infty}^{+\infty} \frac{1}{n^3+1} e^{inx}$$

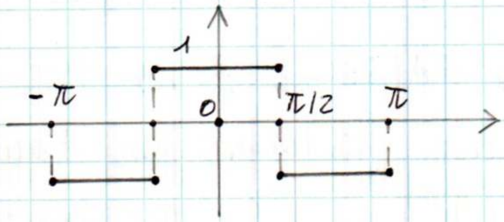


$$f(x) = \sum_{n=-\infty}^{+\infty} \frac{1}{n^{12}+1} e^{inx}$$ is smoother than

$$f(x) = \sum_{n=-\infty}^{+\infty} e^{-|n|} e^{inx}$$ even smoother

Let's compute the coeff. of the F.S. of a couple of functions 2π-periodic, $L^1(-\pi,\pi)$, but with jump discontinuities.

ex.1) "square wave"



$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi/2) \\ -1 & \text{if } x \in (\frac{\pi}{2}, \pi] \\ \text{even, } 2\pi\text{-periodic} \end{cases}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \quad \text{OR} \quad \begin{cases} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overbrace{f(x)}^{\text{EVEN}}\ \overbrace{\sin nx}^{\text{ODD}}}_{\text{ODD}}\, dx = 0 \qquad n = 1, 2, 3, \ldots$$

> In general if our periodic function is EVEN then $b_n = 0$ (only cosine terms). Also if our f is ODD then $a_n = 0$ (only sine terms)

$$f(x) = \underbrace{\sum_{n=1}^{\infty} b_n \sin nx}_{\text{ODD part of } f} + \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}_{\text{EVEN part of } f}$$

$$\frac{f(x) - f(-x)}{2} \qquad + \qquad \frac{f(x) + f(-x)}{2}$$

$$\frac{f(x)}{2} - \cancel{\frac{f(-x)}{2}} + \frac{f(x)}{2} + \cancel{\frac{f(-x)}{2}} = f(x)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overbrace{f(x)}^{\text{EVEN}}\ \overbrace{\cos nx}^{\text{EVEN}}}_{\text{EVEN}}\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx =$$

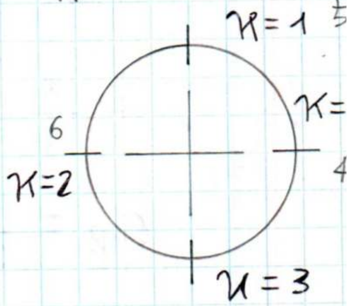
$$= \frac{2}{\pi}\left\{ \int_0^{\pi/2} \cos nx\, dx - \int_{\pi/2}^{\pi} \cos nx\, dx \right\}$$

$a_0 = 0$ if $k \geq 1$ then $\int \cos kx \, dx$

$$= \frac{\sin kx}{k} \text{ so } \int_0^{\pi/2} \cos kx \, dx = \left[\frac{\sin kx}{k} \right]_0^{\pi/2} = \frac{\sin k\frac{\pi}{2}}{k}$$

$$\int_{\pi/2}^{\pi} \cos kx \, dx = \left[\frac{\sin kx}{k} \right]_{\pi/2}^{\pi} = \frac{\cancel{\sin \pi k} - \sin k\pi/2}{k}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx = \frac{2 \cdot 2}{\pi k} \cdot \sin k\frac{\pi}{2}$$

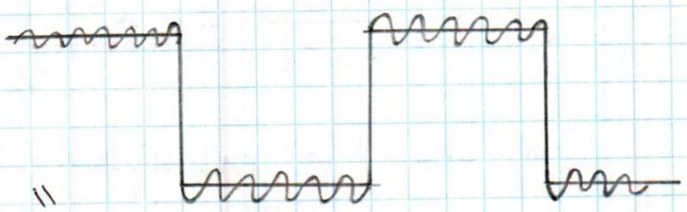


$$a_{2h} = 0 \qquad a_{2h+1} = \frac{2 \cdot 2}{\pi(2h+1)} \underbrace{\sin(2h+1)\frac{\pi}{2}}_{(-1)^h}$$

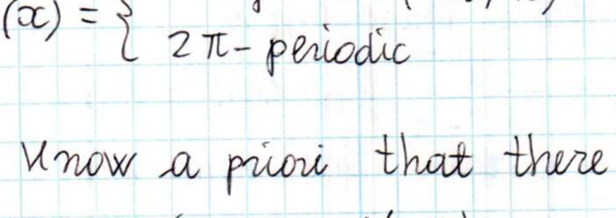
The F.S. is:

$$f(x) = \frac{2}{\pi} \sum_{h=0}^{\infty} \frac{2(-1)^h}{2h+1} \cos[(2h+1)x] = \frac{2 \cdot 2}{\pi} \left\{ \cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x \right.$$

$$\left. - \frac{1}{7}\cos 7x + \dots \right\}$$



ex. 2) $f(x)$ "saw-tooth wave"



f ODD

$$f(x) = \begin{cases} x \text{ if } x \in (-\pi; \pi) \\ 2\pi\text{-periodic} \end{cases}$$

(N.B. ∞ many jump discontinuities at $x = \pi, 3\pi, \dots$ $x = (2k+1)\pi$

We know a priori that there are only sine terms in the F.S. ($a_k = 0 \ \forall k$)

We could compute $b_k = 1/\pi \int_{-\pi}^{\pi} f(x) \sin kx \, dx$ directly (do at home).

We choose instead to compute $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} d$

(then from c_k can obtain the b_k's)

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx$$

$$C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx = 0$$

We need (for $n \neq 0$) the indefinite integral $\int x e^{\alpha x} dx$

by parts

$$\stackrel{\text{by parts}}{=} \int x \, d\left(\frac{e^{\alpha x}}{\alpha}\right) \stackrel{\text{b.p.}}{=\!=} x \frac{e^{\alpha x}}{\alpha} - \int \frac{e^{\alpha x}}{\alpha} dx =$$

$$= \frac{x}{\alpha} \cdot e^{\alpha x} - \frac{e^{\alpha x}}{\alpha^2} = e^{\alpha x}\left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right)$$

$$c_n = \frac{1}{2\pi}\left[e^{-inx}\left(\frac{x}{-in} - \frac{1}{(-in)^2}\right)\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[e^{-i\pi n}\left(\frac{\pi}{-in} + \frac{1}{n^2}\right) +\right.$$

$$\left. - e^{i\pi n} \cdot \left(\frac{-\pi}{-in} + 1/n^2\right)\right] =$$

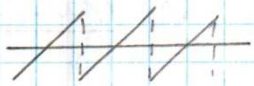
$$\left(\text{by Euler's Formula } e^{i\pi} = e^{-i\pi} = -1\right.$$

$$= \frac{(-1)^n}{\cancel{2}\pi}\left[\frac{\cancel{\pi}}{-in} + \cancel{\frac{1}{n^2}} + \frac{\cancel{\pi}}{-in} - \cancel{\frac{1}{n^2}}\right] =$$

$$= \frac{(-1)^n}{-in} = i \frac{(-1)^n}{n}$$

so the $f(x)$ sawtooth function has this F.S.:

$$f(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} i \frac{(-1)^n}{n} e^{inx}$$



$$\parallel\!\!\backslash \quad \sum_{n=1}^{\infty} b_n \sin nx$$

Exercises for home (write it down)

- compute b_n from these c_n's
- compute $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$
- check that we obtain the same b_n; draw picture with pc.
- Do previous problem $\sum_{n=1}^{\infty} \frac{\sin n}{n}$

Generalized Leibnitz Test

$$\sum_{k=1}^{\infty} a_k b_k$$

(1) $b_k \geq 0$

(2) $\lim_{k \to \infty} b_k = 0$

(3) $b_{k+1} < b_k$

$\left|\sum_{k=1}^{N} a_k\right|$ <u>bounded</u> $\forall N$

in particular the standard Leibnitz test chooses $a_k = (-1)^{\cdot}$

in our exercise $b_k = 1/k$ $\quad a_k = \sin k$

the point was to show that finite sums $\sum_{k=1}^{N} \sin k$ stay bounded in absolute value

<u>Hint</u>: use Euler + G.S.

$$\sin k = \mathrm{Im}\,(e^{ik}) = \mathrm{Im}\,(\cos k + i \sin k)$$

$$\sum_{k=1}^{N} e^{ik} = \sum_{k=1}^{N} (e^{i})^{k} = \frac{1-(e^{i})^{N+1}}{1-e^{i}} - 1 \qquad \left(\text{because } \sum_{k=0}^{N} (e^{ik}) = e^{i0} + \sum_{k=1}^{N} e^{\cdot} \right.$$

$$\left|\sum_{k=1}^{N} e^{ik}\right| = \left|\frac{1-e^{i(N+1)}}{1-e^{i}} - 1\right| \leq \frac{|1-e^{i(N+1)}|}{|1-e^{i}|} + 1 \leq \frac{1+1}{\alpha} + 1$$

$\alpha = |1-e^{i}|$ <u>finite value</u>

$$\sum_{k=1}^{N} \sin k = \mathrm{Im}\left(\sum_{k=1}^{N} e^{ik}\right) \leq \text{FINITE}$$

If it were $\sum_{k=1}^{N} \cos\left(\frac{1}{k}\right)$ instead of $\sum_{1}^{N} \sin k$ it would FAIL

here NOT $\Leftrightarrow$ here we have a cancellation effect

For the sawtooth function we get $c_k = i \cdot \frac{(-1)^k}{k}$

- compute b_k from these c_k's:

The formula we computed was: $b_k = i(c_k - \underline{c}_k)$,

where $\underline{c}_k$ is the complex conjugate of c_k:

$$\underline{c}_k = \overline{c_k} = -i\frac{(-1)^k}{k}$$

Thus we get:

$$b_k = i\left(i\frac{(-1)^k}{k} + i\frac{(-1)^k}{k}\right) = i\left(2i \cdot \frac{(-1)^k}{k}\right) =$$

$$\boxed{b_k = -\frac{2}{k}(-1)^k}$$

- compute $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx \, dx$

$f(x) = x$ if $x \in (-\pi; \pi)$, so:

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx \, dx \underset{\text{by parts}}{=\!=\!=}$$

$$= \frac{1}{\pi}\left[x \cdot \left(-\frac{\cos kx}{k}\right) - \int \left[1 \cdot \left(-\frac{\cos kx}{k}\right)\right] dx\right]\Bigg|_{-\pi}^{\pi} =$$

$$= \frac{1}{\pi}\left[-\frac{x\cos kx}{k} + \frac{1}{k}\int \cos kx \, dx\right]\Bigg|_{-\pi}^{\pi} =$$

$$= \frac{1}{\pi}\left[-\frac{x\cos kx}{k} + \frac{1}{k} \cdot \frac{\sin kx}{k}\right]\Bigg|_{-\pi}^{\pi} =$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos k\pi}{k} + \frac{1}{k^2}\sin k\pi - \left(\frac{\pi\cos(-k\pi)}{k} + \frac{1}{k^2}\sin(-k\pi)\right)\right] =$$

$(\cos(-k\pi) = \cos k\pi$ & $\sin(-k\pi) = -\sin k\pi)$

$$= \frac{2}{\cancel{\pi}}\left(-\frac{\cancel{\pi}}{k}\underbrace{\cos k\pi}_{\approx(-1)^k} + \underbrace{\cancel{\frac{1}{k^2}\sin k\pi}}_{\emptyset}\right)$$

- $\sin k\pi$ is always $\emptyset$
- $\cos k\pi$ is 1 or -1, so it is equal to $(-1)^k$

Finally: $\boxed{b_k = -\frac{2}{k}(-1)^k}$ (Q.D.E.)

The sawtooth wave Fourier Series is:

$$f(x) = -2\sum_{k=1}^{+\infty}\frac{(-1)^k}{k}\sin(kx)$$

$$f(x) = 2\left\{ \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots \right\}$$

Sawtooth function $f(x) = \begin{cases} x & \text{if } x \in (-\pi;\pi) \\ & 2\pi \text{ periodic} \end{cases}$

We have jump discontinuities at the points $x = (2k+1)\pi$
$k \in \mathbb{Z}$

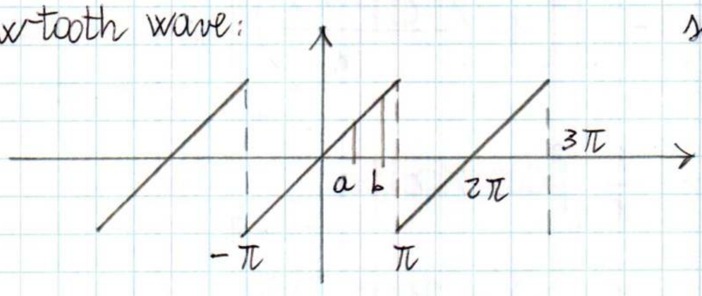
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx \, dx \qquad k = 1,2,3,\dots$$

$a_k = 0$ for $k = 0,1,2,3,\dots$ because $f(x) = -f(-x)$ ODD FUNCTION

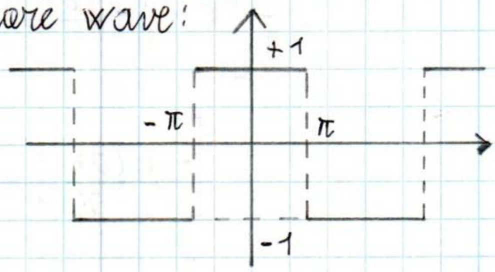
We know that, in general, <u>if</u> f is T-periodic (usually, for convenience, $T = 2\pi$) and $f \in L^1(-\frac{T}{2};\frac{T}{2})$ $\left(\Rightarrow f \in L^2(-\frac{T}{2};\frac{T}{2})\right)$ <u>then</u> $\lim_{N\to\infty} \| S_N(x) - f(x) \|_2 = 0$

<u>Theorem</u> If f (as before) is also $C^1([a;b])$, with $[a;b] \subset [-\frac{T}{2};\frac{T}{2}]$
$\Rightarrow S_N(x) \xrightarrow{\text{POINTWISE}} f(x)$ for $x \in [a;b]$

sawtooth wave:



square wave:



For exercise, let's check that the square wave F.S. converges to the correct value $\equiv 1$ at $x = 0$. We get

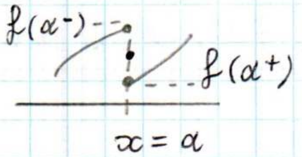
$$\frac{4}{\pi}\left\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right\} \quad \text{but} \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4} \quad \text{so OK}$$

To show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$ start from $\frac{1}{1-t} \overset{G.S.}{=} 1 + t + t^2 + \dots$
for $|t| < 1$, $t = -x^2$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$ (because of G.S.), then integrate:

$$\int_{-\pi}^{\pi} \frac{1}{1+x^2}\,dx = \int_{-\pi}^{\pi} (1 - x^2 + x^4 - x^6 + \dots)\,dx \rightarrow \text{result,}$$
CFR. page 9R

Theorem If f is T-periodic and $L^1(-\frac{T}{2};\frac{T}{2})$ and has jump discontinuity at $x=\alpha$

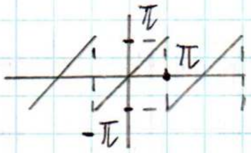


let $f(\alpha^-)=\lim\limits_{x\to\alpha^-} f(x)$ (finite value)

and $f(\alpha^+)=\lim\limits_{x\to\alpha^+} f(x)$ (finite value)

Assume $\underset{\lim\limits_{x\to\alpha^-} f'(x)}{\underset{|||}{f'(\alpha^-)}}$ exists finite and $\underset{\lim\limits_{x\to\alpha^+} f'(x)}{\underset{|||}{f'(\alpha^+)}}$ exists finite $\Rightarrow$ the F.S. of f converges to $\dfrac{f(\alpha^-)+f(\alpha^+)}{2}$ (MIDPOINT OF JUMP)

Let's check this theorem with $\underset{\text{SAWTOOTH}}{f(x)}=2\left\{\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\dots\right\}=2\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$



At $x=\pi$ we have $f(\pi^-)=\pi$ and $f(\pi^+)=-\pi$

Also $\begin{cases} f'(\pi^-)=1 \\ f'(\pi^+)=1 \end{cases}$

(4 finite values, so assumptions of the th. are OK) MID-JUMP $=$

$=\dfrac{f(\pi^-)+f(\pi^+)}{2}=\dfrac{\pi-\pi}{2}=0$, and if we plug $x=\pi$ in F.S.

$0=2\left\{\cancel{\sin\pi}-\frac{1}{2}\cancel{\sin 2\pi}+\frac{1}{3}\cancel{\sin 3\pi}-\frac{1}{4}\cancel{\sin 4\pi}+\cancel{\dots}\right\}=0$ (CVD)

We have seen that a <u>linear ODE</u> (of order 2 in this example)

$a_2y''+a_1y'+a_0y=f(x)$ has a general solution of the form:

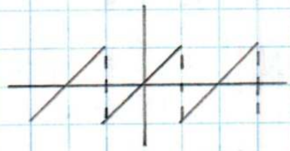
$\quad\quad\quad\quad\quad\quad \hookrightarrow$ <u>FORCING TERM</u>

$y_{GEN}(x)=c_1y_1(x)+c_2y_2(x)+\eta(x)$, where $c_1y_1(x)+c_2y_2(x)$ is the solution of $a_2y''+a_1y'+a_0y=0$ (<u>associated homogeneous ODE</u>) and $\eta(x)$ is <u>any solution of the non-homogeneous ODE</u> (for constant coeff. a_2, a_1, a_0 the we solve $a_2t^2+a_1t+a_0=0$ and obtain $y_1(x)$ and $y_2(x)$)

If $f(x)$ is T-periodic (let's say $T=2\pi$) we can write $f(x)$ with a F.S. and we can find $\eta(x)$ with another F.S.

Example:

$$y'' - 7y' + 10y = \underset{\text{SAWTOOTH}}{f(x)}$$



$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$$= i\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n} e^{inx} \quad \text{(complex form)}$$

Let's solve $y'' - 7y' + 10y = 0$ (lin. hom. assoc. ODE)

$$t^2 - 7t + 10 = 0 \quad t \begin{cases} 2 \\ 5 \end{cases}$$

$$\underset{\text{HOM}}{y(x)} = c_1 e^{2x} + c_2 e^{5x}$$

$$\eta(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$ (here we are assuming that, since $f(x)$ is 2π-periodic, also $\eta(x)$ will be)

then plugging this F.S. expression of $\eta(x)$ in the full ODE we get:

$$\eta' = \sum_{n=-\infty}^{+\infty} inc_n e^{inx}$$

$$\eta'' = \sum_{n=-\infty}^{+\infty} -n^2 c_n e^{inx}$$

$$\sum_{n=-\infty}^{+\infty} -n^2 c_n e^{inx} - 7\sum_{-\infty}^{+\infty} inc_n e^{inx} + 10\sum_{-\infty}^{+\infty} c_n e^{inx} =$$

$$= i\sum_{-\infty}^{+\infty} \frac{(-1)^n}{n} e^{inx}$$

Collecting out $c_n e^{inx}$:

$$\sum_{n=-\infty}^{+\infty} \underbrace{(-n^2 - 7in + 10)c_n} e^{inx} = \sum_{n=-\infty}^{+\infty} \underbrace{\frac{(-1)^n i}{n}} e^{inx}$$

<u>Fact</u> 2 F.S. coincide $\Leftrightarrow$ the amplitude corresponding to each frequency coincides.

$$(-n^2-7in+10)c_n = i\,\frac{(-1)^n}{n} \quad \text{for } n=\dots,-3,-2,-1,0,1,2,\dots$$

$$c_n = \frac{i(-1)^n}{n(-n^2-7in+10)}$$ We found $\eta(x)$ as a F.S.

Substitute in $\eta(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$

$$y_{TOT}(x) = c_1 e^{2x} + c_2 e^{5x} + \sum_{n=-\infty}^{+\infty} \frac{i(-1)^n}{n(-n^2-7in+10)} e^{inx}$$

Rem. Since $f(x)$ and the coeff. of our ODE were real, also the solution will be real.

$\Rightarrow$ computing a_n and b_n from c_n, all imaginary terms must disappear.

Alternatively we could directly write

$$\eta(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$ $\Big\{$ we woul have $a_n=$

$$\boxed{\text{One should resist the temptation to assume that } \eta(x) \text{ is odd because } f(x) \text{ is odd.}}$$

In fact, the derivative of an odd function is even & the derivative of an even function is odd, and we plug η, η', η'' in the ODE.

$$\eta'(x) = \sum_{n=1}^{\infty} (-n a_n \sin nx + n b_n \cos nx)$$

$$\eta''(x) = \sum_{n=1}^{\infty} (-n^2 a_n \cos nx - n^2 b_n \sin nx)$$

$$y'' - 7y' + 10y = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$$\sum_{n=1}^{\infty} (-n^2 a_n \underbrace{\cos nx} - n^2 b_n \overbrace{\sin nx}) - 7\sum_{n=1}^{\infty} (-n a_n \overbrace{\sin nx} + n b_n \underbrace{\cos nx})$$

$$+ \underbrace{5a_0}_{\overset{|||}{0}} + 10\sum_{n=1}^{\infty} (a_n \underbrace{\cos nx} + b_n \overbrace{\sin nx}) \equiv 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$a_0 = 0$ Then we collect $\cos nx$ and $\sin nx$ =