$$\sum_{k=1}^{\infty} \underbrace{(-k^2 a_k - 7k b_k + a_k \cdot 10)}_{\text{must be "0" for } k=1,2,3,\dots} \cos kx + (-k^2 b_k + 7k a_k + 10 b_k) \sin kx = \sum_{k=1}^{\infty} \frac{2}{k} (-1)^{k+1} \sin kx$$

$$-k^2 b_k + 7k a_k + 10 b_k = \frac{2}{k} (-1)^{k+1}$$

In this case it was better the complex computation.

F.S. are a good tool when in an ODE (and later, as we'll see, in PDE) we need to represent a known and/or unknown periodic function.

Also F.S. are a good tool when we are interested on functions on a finite interval $[a,b]$ (because we can always assume that $f(x)$ for $x \in [a,b]$ is one period of a periodic function). They are NOT a good tool for functions on $(-\infty; +\infty)$ or $(0; +\infty)$.

Quick introduction on **FOURIER TRANSFORM**

Def. The Fourier Transform $\hat{f}$ of a function f is given by the integral

$$\boxed{\hat{f}(t) = \int_{-\infty}^{+\infty} f(x) e^{-itx}\, dx = \int_{-\infty}^{+\infty} f(x) (\cos tx - i \sin tx)\, dx}$$

this (improper) integral is well defined (in the classical sense) if $f \in L^1(\mathbb{R})$ (f is such that $\int_{-\infty}^{+\infty} |f(x)|\, dx$ is FINITE). There are generalizations of this definition (... later).

N.B. On the line $L^2(\mathbb{R}) \not\subset L^1(\mathbb{R})$ and $L^1(\mathbb{R}) \not\subset L^2(\mathbb{R})$ (unlike $L^1([a,b])$ which contains all $L^p([a,b])$ for $p>1$, in particular $L^2([a,b])$) [CFR page 18]

If $f \in L^2(\mathbb{R})$, i.e., $\|f\|_2 = \left(\int_{-\infty}^{+\infty} |f(x)|^2 dx\right)^{1/2}$ is <u>finite</u>, then we <u>define</u> $\hat{f}(t) = \lim_{N \to \infty} \int_{-N}^{N} f(x) e^{-ixt} dx$

$$\boxed{\text{In fact, the F.T. maps } L^2(\mathbb{R}) \to L^2(\mathbb{R})\text{, this map is bijective (injective and surjective) and it's also isometric } (\|\hat{f}\|_2 = c\|f\|_2)}$$

<u>Prop.</u> If f is <u>even</u> & <u>real-valued</u> $\Rightarrow$ $\boxed{\hat{f}(t) = 2\int_0^{+\infty} f(x) \cos xt \, dx}$

<u>Proof:</u> $$\hat{f}(t) = \int_{-\infty}^{+\infty} \underbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\cos xt}_{\text{even}}}_{\text{even}} - i \underbrace{\int_{-\infty}^{+\infty} \underbrace{\underbrace{f(x)}_{\text{EVEN}} \underbrace{\sin xt}_{\text{ODD}}}_{\text{ODD}} dx}_{0} =$$

$$= 2\int_0^{+\infty} f(x) \cos xt \, dx$$

If f is <u>ODD</u> and real-valued, then $\hat{f}$ is odd and pure-imaginary $\boxed{\hat{f}(t) = -2i \int_0^{+\infty} f(x) \sin xt \, dx}$

<u>Proof</u>: $$\hat{f}(t) = \underbrace{\int_{-\infty}^{+\infty} \underbrace{\underbrace{f(x)}_{\text{ODD}} \underbrace{\cos xt}_{\text{EVEN}}}_{\text{ODD}}}_{0} - i \int_{-\infty}^{+\infty} \underbrace{\underbrace{f(x)}_{\text{ODD}} \underbrace{\sin xt}_{\text{ODD}}}_{\text{EVEN}} =$$

$$= -2i \int_0^{+\infty} f(x) \sin xt \, dx$$

Beware!!
with integers
ODD · EVEN = EVEN
with functions
ODD · EVEN = ODD
$f(x) = f(-x)$
$f(x) = -f(-x)$

<u>Linearity</u>

$$\boxed{[\alpha f(x) + \beta g(x)]^{\wedge}(t) = \alpha \hat{f}(t) + \beta \hat{g}(t)}$$

(Proof obvious) (easy but important fact)

(The transform of a linear combination of functions is equal to the linear combination of the transforms)

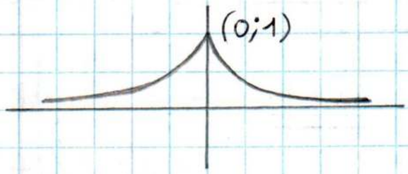
$$[f'(x)]^\wedge(t) = (it)\,\hat{f}(t)$$

$$[f''(x)]^\wedge(t) = (it)^2\,\hat{f}(t)$$

$$\vdots$$

$$[f^{(n)}(x)]^\wedge(t) = (it)^n\,\hat{f}(t)$$

<ins>**ex.**</ins> Compute the F.T. of $f(x) = e^{-|x|}$



clearly $f \in L^1(\mathbb{R})$

$$\hat{f}(t) = \int_{-\infty}^{+\infty} e^{-|x|} e^{-ixt}\,dx$$

$$\begin{cases} e^{-x} & \text{if } x \geq 0 \\ e^{x} & \text{if } x \leq 0 \end{cases}$$

<ins>N.B.</ins> The F.T. of $f(x) = e^x$ is not well defined (at least in the classical way) because $e^x \notin L^1(\mathbb{R})$

$$\hat{f}(t) = \int_{-\infty}^{+\infty} e^{-|x|} e^{-ixt}\,dx = \int_0^{+\infty} \underbrace{e^{-x}e^{-ixt}}_{e^{-x(1+it)}}\,dx + \int_{-\infty}^{0} \underbrace{e^{x}e^{-ixt}}_{e^{x(1-it)}}\,dx$$

$$= \lim_{b\to+\infty} \left[\frac{e^{-x(1+it)}}{-(1+it)}\right]\Bigg|_0^b + \lim_{a\to-\infty}\left[\frac{e^{x(1-it)}}{1-it}\right]\Bigg|_a^0 =$$

$$= \frac{-1}{1+it} \lim_{b\to+\infty}\Big(\underbrace{e^{-b(1+it)}}_{\Downarrow\, 0} - 1\Big) + \frac{1}{1-it}\lim_{a\to-\infty}\Big(1 - \underbrace{e^{a(1-it)}}_{\Downarrow\, 0}\Big) =$$

$$= \frac{1}{1+it} + \frac{1}{1-it} = \frac{1-it+1+it}{(1+it)(1-it)} = \frac{2}{1+t^2}$$

$f(x)$	$\hat{f}(t)$
$e^{-\lvert x\rvert}$	$\frac{2}{1+t^2}$

Notation we will write both $F[f(x)]$ or $\hat{f}(t)$ or $[f(x)]^{\wedge}(t)$ for the Fourier Transform of f.

$$F[f(x-y)](t) = e^{-ity}\hat{f}(t)$$

(multiplication by a complex exponential is called modulation)

[F.T. maps translations into modulations]

$$F[e^{ixy}f(x)](t) = \hat{f}(t-y)$$

[& vice-versa]

$$F\left[f\left(\frac{x}{\lambda}\right)\right] = \lambda\hat{f}(\lambda t)$$

(horizontal rescaling is mapped into reverse-horizontal and vertical rescaling)

Riemann - Lebesgue Theorem

If $f \in L^1(\mathbb{R})$, **then** $\hat{f} \in C_0(\mathbb{R})$, i.e., the function $\hat{f}(t)$ is continuous for $t \in \mathbb{R}$, **and** $\lim_{t\to\pm\infty}\hat{f}(t) = 0$.

Furthermore $\|\hat{f}\|_\infty = \sup_{t\in\mathbb{R}}|\hat{f}(t)|$

and $\|f\|_1 = \int_{-\infty}^{+\infty}|f(x)|\,dx$ satisfy $\|\hat{f}\|_\infty \leq \|f\|_1$

N.B. The map $F: L^1(\mathbb{R}) \to C_0(\mathbb{R})$ **is not** surjective (it is **injective**)

There are functions $g \in C_0(\mathbb{R})$ which are not images of $f \in L^1(\mathbb{R})$ under F.T.

It is possible to show that $F: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ **bijectively** (Plancherel's theorem).

Compare this with $F: L^1(\mathbb{R}) \to C_0(\mathbb{R})$: the map is not surjective and it is difficult to describe $F(L^1(\mathbb{R})) \subset C_0(\mathbb{R})$

Inversion Formula

$$\boxed{f(x) = F^{-1}[\hat{f}] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(t)\, e^{ixt}\, dt \qquad (*)}$$

Remarks

(1) the inversion formula looks very similar to the direct formula (3 differences:
- factor $1/2\pi$ in front of the integral;
- e^{ixt} factor instead of e^{-ixt};
- the integral is in t.)

(2) Not all functions in $C_0(\mathbb{R})$ are also in $L^1(\mathbb{R})$, so the $(*)$ inverse transform is defined in classical way only in a subset of cases...

(3) the inversion formula, when valid, is equivalent to:

$$\boxed{F[F[f]](x) = 2\pi f(-x)}$$

(4) in the example we computed $f(x) = e^{-|x|} \in L^1(\mathbb{R})$, $\hat{f}(t) = \frac{2}{1+t^2} \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$, so there is no problem in applying the inversion formula

i.e. $\frac{1}{\cancel{2}\pi}\int_{-\infty}^{+\infty} \frac{\cancel{2}}{1+t^2} e^{ixt}\, dt = e^{-|x|}$

$$\begin{tikzcd} f(x) = e^{-|x|} \arrow[r, bend left, "F"] & \frac{2}{1+t^2} \arrow[l, bend left, "F^{-1}"] \end{tikzcd}$$

Both F and F^{-1} are operators (function between functions)

(5) This formula $(*)$ can be thought as a way of representing $f(x)$ $(x \in \mathbb{R})$ as a superposition of waves $e^{ixt} = \cos xt + i \sin xt$

sinusoidal of frequency depending on $t \in \mathbb{R}$ and "amplitude" $\hat{f}(t)$. Please compare with the F.S. (complex version) of

$f(x)$ 2π-periodic $\quad f(x) = \sum_{k=-\infty}^{+\infty} c_k e^{ikx}$ (discrete frequencies amplitudes c_k)

e^{ikx} analog of e^{ixt} (t analog k)

$\sum_{k=-\infty}^{+\infty} \dots$ analog of $\int_{-\infty}^{+\infty} \dots dt$

(in (*) we have continuous frequencies)

Note that we have a factor $1/2\pi$ in front of the I.F.T. and no factor in front of the F.T.

Depending on the book you look at, sometimes other conventions are used:

Alternatives: $\frac{1}{\sqrt{2\pi}}$ in front both of F.T. and I.F.T.

Also factor 1 in front of both F.T. and I.F.T. but

$$\hat{f}(t) = \int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i x t}\, dx$$

In the book puts $1/2\pi$ in front of F.T., 1 for I.F.T.

Derivative properties of F.T.

$$\boxed{F[f'(x)] = (it)\,\hat{f}(t)}$$

$$\boxed{F[x f(x)] = i\,\hat{f}'(t)}$$

Plancherel / Parseval identities

Assume that $\|f\|_2 = \left(\int_{-\infty}^{+\infty} |f(x)|^2 dx\right)^{1/2}$ is finite

then also $\|\hat{f}\|_2$ is finite and $\boxed{\|\hat{f}\|_2 = \sqrt{2\pi}\,\|f\|_2}$

$$\boxed{\int_{-\infty}^{+\infty} f(y)\,\hat{g}(y)\,dy = \int_{-\infty}^{+\infty} \hat{f}(y)\, g(y)\, dy} \quad [\text{Parseval}]$$

Def. Convolution of two functions (SEE ALSO PAGE 31R)

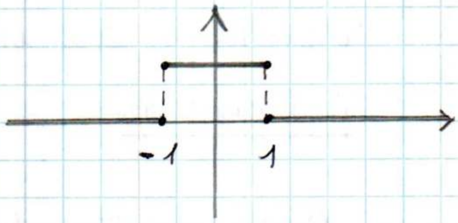
$$\underbrace{(f * g)(x)}_{f \text{ STAR } g \text{ or CONVOLUTION of } f \text{ AND } g} \overset{\text{def.}}{=\!=} \int_{-\infty}^{+\infty} f(x-y)\cdot g(y)\, dy$$

Good news: $\boxed{F[f*g] = \hat{f}(t)\cdot\hat{g}(t)}$

(The F.T. maps convolution products into products)

Exercises Compute these transforms:

ex. F.T. of $f(x) \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$



$$\hat{f}(t) = \int_{-\infty}^{+\infty} f(x)\, e^{-ixt}\, dx = \int_{-1}^{1} e^{-ixt}\, dx =$$

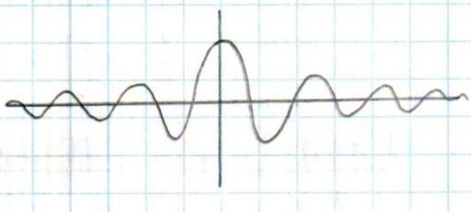
'cause the function is zero outside this interval

$$= \left[\frac{e^{-ixt}}{-it} \right]\Bigg|_{-1}^{1} =$$

$$= \frac{1}{-it}\left(e^{-it} - e^{it}\right) = \frac{e^{it} - e^{-it}}{it}\cdot\frac{2}{2} = 2\,\frac{\sin t}{t}$$

$\Downarrow$ sin t

$f \in L^1(\mathbb{R})$ not C_0



$\hat{f} \in C_0(\mathbb{R})$

f even R.V. (symmetric with respect to y axis)

$\hat{f}$ even R.V. (" " " " " ")

$\hat{f}(t) = 2\,\frac{\sin t}{t}$ Plancherel $\boxed{\|\hat{f}\|_2^2 = 2\pi\, \|f\|_2^2}$

with this formula we can compute the highly not-trivial integral $\sin^2 t / t^2$

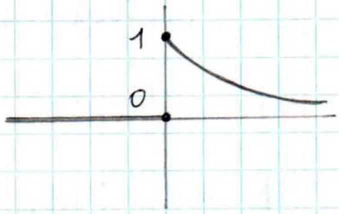
$$\|f\|_2^2 = \int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-1}^{1} 1\, dx = 2 \qquad 2\pi\|f\|_2^2 = 4\pi$$

$$\int_{-\infty}^{+\infty} \left(\cancel{2}\frac{\sin t}{t}\right)^2 dt = \cancel{4}\pi \qquad \int_{-\infty}^{+\infty} \frac{\sin^2 t}{t^2}\, dt = \pi$$

$$\boxed{\int_{-\infty}^{+\infty} \frac{\sin^2 t}{t^2}\, dt = \pi}$$

(20 € if you can compute this in a difficult way)

Another ex.) $f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$



$$\hat{f}(t) = \int_{-\infty}^{+\infty} f(x)\, e^{-ixt}\, dx =$$

$$= \int_0^{+\infty} e^{-x} e^{-ixt}\, dx = \int_0^{+\infty} e^{-x(1+it)}\, dx =$$

$$= \lim_{b \to +\infty} \left[\frac{e^{-x(1+it)}}{-(1+it)} \right] \Bigg|_0^b = \frac{1}{1+it}$$

Remark

$$\left(e^{-|x|}\right) \xrightarrow{F} \frac{2}{1+t^2}$$ quadratic decay as $t \to \pm\infty$

$$\begin{pmatrix} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{pmatrix} \longrightarrow \frac{1}{1+it}$$ degree 1 decay at ∞

We are observing a special case of a F.T. property:

$$\left(\text{smoothness of } f\right) \longrightarrow \left(\text{decay as } t \to \pm\infty \text{ of } \hat{f}(t)\right)$$

and this is a case of the LOCAL/GLOBAL principle that we will study later. (Page 30R - 31)

[CFR ALSO P.18 bottom]

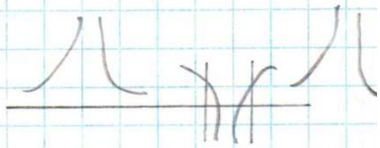
LESS. 5 06/11/2008

Remark about integrals on the line:

$$\int_{-\infty}^{+\infty} f(x)\,dx$$

When we say $f \in L^1(\mathbb{R})$ we mean that $\int_{-\infty}^{+\infty} |f(x)|\,dx$ exists and is finite.

The $\int_{-\infty}^{+\infty} \dots dx \overset{\text{def.}}{=} \lim\limits_{b\to+\infty} \int_{-b}^{b} \dots dx$

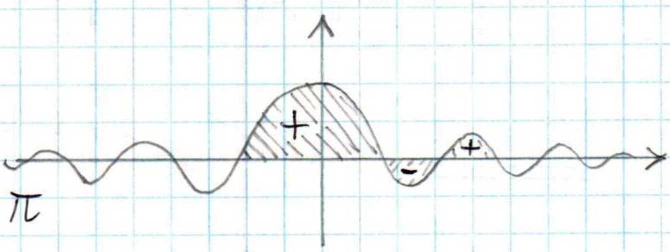


(actually if $f(x) \to \pm\infty$ at a finite value $x = a$ or many of those we have to take extra limits)

<u>N.B.</u> There are cases where $\lim\limits_{b\to+\infty} \int_{-b}^{b} f(x)\,dx$ exists finite BUT

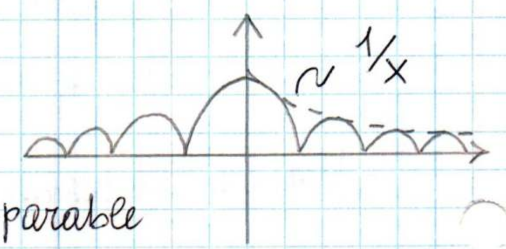
$$\lim_{b\to+\infty} \int_{-b}^{b} |f(x)|\,dx = +\infty$$

<u>example</u>: $f(x) = \dfrac{\sin x}{x}$



we have $\lim\limits_{b\to+\infty} \int_{-b}^{+b} \dfrac{\sin x}{x}\,dx = \pi$

<u>but</u> $\lim\limits_{b\to+\infty} \int_{-b}^{+b} \left|\dfrac{\sin x}{x}\right| dx = +\infty$



The area of $\int_{1}^{+b} \left|\dfrac{\sin x}{x}\right| dx$ is comparable to $\int_1^b 1/x\,dx$ (i.e. there are 2 constants $c>0$; $d>0$ such that $c\int_1^b 1/x\,dx < \int_1^b \left|\dfrac{\sin x}{x}\right| dx < d\int_1^b 1/x\,dx$)

and $\lim\limits_{b\to+\infty} \int_1^b 1/x\,dx = \lim\limits_{b\to+\infty} (\log b - \log 1) = +\infty$

The same phenomenon happens with <u>series</u>, for example

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \log 2 \quad \text{but} \quad \sum_{n=0}^{+\infty} \left|\frac{(-1)^n}{n+1}\right| = +\infty$$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

$\int \frac{\sin x}{x}\,dx$ <u>is not</u> "elementary", because the function

$F(x)' = \frac{\sin x}{x}$ is a special function.

One possible tool is power series:

$\sin x = x - x^3/3! + x^5/5! - \dots$

$\frac{\sin x}{x} = 1 - x^2/3! + x^4/5! - \dots \quad (R=\infty)$

$$\int_a^b \frac{\sin x}{x}\,dx = \left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \dots \right]_a^b$$

this gives a series representation of the integral over $[a;b]$ <u>bounded</u> interval, but it's tricky to use as $a \to -\infty$ and $b \to +\infty$ (plus, ideally, we aim at "exact" values like π).

$$f(x) = \begin{cases} 1 & \text{for } x \in [-1;1] \\ 0 & \text{for } x \notin [-1;1] \end{cases} \Rightarrow \hat{f}(t) = 2\frac{\sin t}{t}$$

$$\hat{f}(t) = \int_{-\infty}^{+\infty} e^{-ixt} f(x)\,dx \qquad \text{(direct formula)}$$

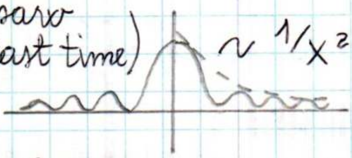
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(t)\, e^{ixt}\,dt \qquad \text{(\underline{inversion formula})}$$

in particular $f(0) = 1 = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{+\infty} \cancel{2}\,\frac{\sin t}{t}\,dt$

$$\Rightarrow \boxed{\pi = \int_{-\infty}^{+\infty} \frac{\sin t}{t}\,dt}$$

Using Plancherel's Theorem we can also compute exactly

$\int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^2 dx$ (we saw it last time)



$\sim 1/x^2$

N.B. $\left(\frac{\sin x}{x}\right)^2 \in L^1(\mathbb{R})$

<u>Moral</u> $\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx$ <u>is not</u> absolutely convergent

$\int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^2 dx$ <u>is</u> absolutely convergent.

An easy way to compute both of them is via Fourier transforms (another technique would be to use complex analysis, residues, contour integration ...).

<u>Remark</u> $\boxed{\hat{f}(t) = \lim_{N\to\infty} \int_{-N}^{N} f(x) e^{-ixt} dx}$

If $f \in L^1(\mathbb{R})$ but also if $f \in L^2(\mathbb{R})$ this limit is well defined.
This definition could be generalized using the theory of <u>distributions</u> (e.g. Dirac δ δ' δ'' more...)

<u>Remark</u> $F(f)_{(t)} = \hat{f}(t)$ FOURIER TRANSFORM (OPERATOR: function between functions)

$F: L^2(\mathbb{R}) \to L^2(\mathbb{R})$
injective, surjective, <u>isometric</u> map between function spaces

$\downarrow$ $\|\hat{f}\|_2 = \sqrt{2\pi} \cdot \|f\|_2$ (Plancherel)

(in many cases you can map difficult integrals to easy ones)

<u>Some remarks on L^2 geometry.</u>
In $\mathbb{R}^n = \left\{ \vec{x} = (x_1, x_2, \dots, x_n) \text{ with } n \text{ real components} \right\}$ there is a geometric notion of <u>distances</u> and of <u>angles</u>, obtainable from the definition of <u>scalar product</u> (in particular if $n=2$, or $n=3$ we get the usual Euclidean geometry)

<u>**Def.**</u> If $\vec{x}, \vec{y} \in \mathbb{R}^n$ $\vec{x} \cdot \vec{y} = \sum_{k=1}^{n} x_k y_k = x_1 y_1 + x_2 y_2 + \dots + x_n$

SCALAR PRODUCT
$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ } maps vectors into scalars

In particular $\vec{x}\cdot\vec{x} = x_1^2+x_2^2+\ldots+x_n^2$

and $\boxed{|\vec{x}| = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{x_1^2+x_2^2+\ldots+x_n^2}}$

We'll see that Plancherel theorem is an infinite-dimensional pithagorean theorem.

Distance from $\vec{x}$ to $\vec{y} \stackrel{def.}{=} |\vec{x}-\vec{y}|$

Theor. $\boxed{\vec{x}\cdot\vec{y} = |\vec{x}|\cdot|\vec{y}|\cos\alpha}$

in particular $\vec{x}\perp\vec{y} \iff \vec{x}\cdot\vec{y}=0$

We can complexify the scalar product extending $\mathbb{R}^n$ to $\mathbb{C}^n = \{\vec{v}=(z_1, z_2, \ldots, z_n)$, with $z_k = x_k + i y_k \in \mathbb{C}$ for $k=1,2,\ldots,n\}$

Def. of scalar product in $\mathbb{C}^n$:

if $\vec{v},\vec{w}\in\mathbb{C}^n$, then $\vec{v}\cdot\vec{w} = \sum_{k=1}^{n} v_k \underset{\text{conjugate}}{\overline{w_k}} =$

$= v_1\overline{w_1} + v_2\overline{w_2} + \ldots + v_n\overline{w_n}$

N.B. if in particular $\vec{v}\in\mathbb{R}^n\subset\mathbb{C}^n$ and $\vec{w}\in\mathbb{R}^n\subset\mathbb{C}^n$ we get the same def.'s as before.

$\text{dist}(\vec{v},\vec{w}) = |\vec{v}-\vec{w}| \qquad \vec{v},\vec{w}\in\mathbb{C}^n$

$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\alpha$

where, e.g. $|\vec{v}| = v_1\bar{v}_1 + v_2\bar{v}_2 + \ldots + v_n\bar{v}_n$

$= |v_1|^2 + |v_2|^2 + \ldots + |v_n|^2$

$(z\bar{z} = (x+iy)(x-iy) = x^2 + \cancel{ixy} - \cancel{ixy} + y^2 =$

$= x^2+y^2 = |z|^2)$

We can define a (complex or real) scalar product on $L^2(\mathbb{R})$ (could actually be $L^2(A)$ with $A\subseteq\mathbb{R}$)

(f,g) SCALAR PRODUCT of $f(x)$ & $g(x)$

$$\boxed{(f, g) = \int_A f(x)\, \overline{g(x)}\, dx}$$ SCALAR PRODUCT on $L^2(A)$ of $f(x)$ & $g(x)$

In particular, if we take $g = f$ we get:

$$(f, f) = \int_A f(x)\overline{f(x)}\, dx = \int_A |f(x)|^2 dx$$

So $\boxed{(f, f) = \|f\|_2^2}$, where $\|f\|_2 = \|f\|_{L^2(A)} = \left(\int_A |f(x)|^2 dx\right)^{1/2}$

$$\boxed{(f, g) = \|f\|_2 \|g\|_2 \cos\alpha}$$

We say that 2 functions f & g are orthogonal to each other in $L^2(A)$ and we write $f \perp g$ if

$$\boxed{\int_A f(x)\, \overline{g(x)}\, dx = 0}.$$

For example $f_k(x) = e^{ikx}$ for $k \in \mathbb{Z}$ and consider $L^2(A)$ with $A = [-\pi; \pi]$, then $(f_k, f_h) = 0$ if $k \neq h$

$= 2\pi$

we have proved this

F.T. Properties

(1) $\boxed{F[f(x-y)]_{(t)} = e^{-ity} \hat{f}(t)}$ [translations are mapped into modulations]

Proof $\;\;\equiv \int_{-\infty}^{+\infty} f(x-y)\, e^{-ixt} dx =$

change of variable $x = x' + y$

$$= \int_{-\infty}^{+\infty} f(x')\, e^{-i(x'+y)t} dx' =$$

$$= e^{-iyt} \underbrace{\int_{-\infty}^{+\infty} f(x)\, e^{-ixt} dx}_{\hat{f}(t)} \quad \text{(QED)}$$

(2) $\boxed{F[e^{ixy} f(x)]_{(t)} = \hat{f}(t-y)}$ [viceversa]

Proof $\quad$ l.h.s. $= \int_{-\infty}^{+\infty} e^{ixy} f(x)\, e^{-ixt} dx =$

$$= \underbrace{\int_{-\infty}^{+\infty} f(x)\, e^{-ix(t-y)}\, dx}_{\hat{f}(t-y)} \quad \text{(QED)}$$

(3) $\boxed{F[f(\tfrac{x}{\lambda})]_{(t)} = \lambda \hat{f}(\lambda t)}$ (horizontal rescaling is mapped into reverse-horizontal and vertical rescaling)

Proof $\quad$ l.h.s. (left hand side) $= \int_{-\infty}^{+\infty} f(x/\lambda)\, e^{-ixt}\, dx \qquad \underline{\lambda > 0}$

$x = \lambda x'$

$$= \int_{-\infty}^{+\infty} f(x')\, e^{-i\lambda x' t} \cdot \lambda\, dx' = \lambda \int_{-\infty}^{+\infty} f(x)\, e^{-i(\lambda t)x}\, dx$$

$$= \lambda \hat{f}(\lambda t) \quad \text{(QED)}$$

(4) $\boxed{F[f'(x)]_{(t)} = it\, \hat{f}(t)}$ [Derivative properties]

Proof $\quad$ l.h.s. $= \int_{-\infty}^{+\infty} f'(x)\, e^{-ixt}\, dx \overset{\text{by parts}}{=} \underbrace{\left[f(x)\, e^{-ixt} \right]_{-\infty}^{+\infty}}_{0} +$

$$- \int_{-\infty}^{+\infty} f(x)(-it)\, e^{-ixt}\, dx$$

Since f is in L^1 or $L^2(\mathbb{R})$, we can assume $\lim_{x \to \pm\infty} f(x) = 0$

by Euler's formula $e^{-ixt} = \cos xt - i \sin xt$, which are bounded functions of x for any fixed $t \in \mathbb{R}$.

By calculus $\lim_{x \to \pm\infty} f(x)\, e^{-ixt} = 0$

The remaining term is $it \int_{-\infty}^{+\infty} f(x)\, e^{-ixt}\, dx = it\, \hat{f}(t)$

(QED)

Corollary (for n-th derivatives)

$$\boxed{F[f^{(n)}(x)]_{(t)} = (it)^n \hat{f}(t)}$$

Fourier transform maps derivatives into product with powers.

$$\boxed{F\,[x f(x)]_{(t)} = i\,(\hat{f}(t))'} \qquad \text{(inverse relation)}$$

<u>Proof</u> $\quad r.h.s. = i\left[\int_{-\infty}^{+\infty} f(x)\, e^{-ixt}\,dx\right]'$

(we should justify the derivative d/dt under the integral)

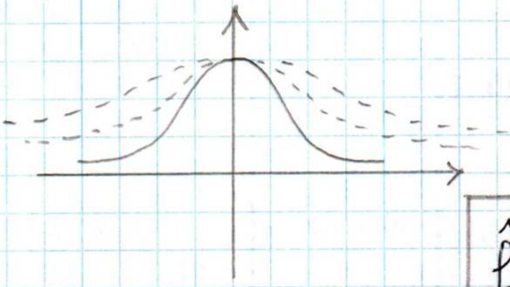
$$= i\int_{-\infty}^{+\infty} f(x)\,(-ix)\,e^{-ixt}\,dx = \int_{-\infty}^{+\infty} x f(x)\, e^{-ixt}\,dx =$$

$$= \underbrace{F\,[x f(x)]}_{= \ l.h.s.} \qquad \text{(QED)}$$

Gaussian Functions

(*) $\quad \boxed{f(x) = e^{-ax^2}} \qquad \underline{a > 0}$

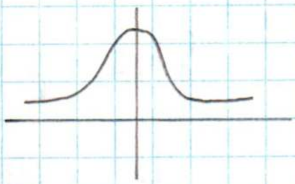
"flattens" as $a \to 0$

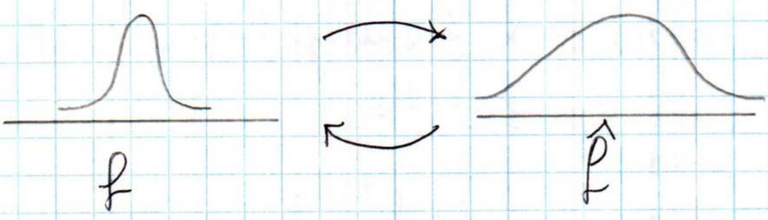


"bell shaped"

important in probability, quantum mechanics and many other fields

$$\boxed{\hat{f}(t) = \sqrt{\frac{\pi}{a}}\; e^{-\frac{t^2}{4a}} \text{ is still a bell curve}}$$





we'll go back to this with the "uncertainty principle" and "local/global principle"

from sharper to flatter & vice-versa!

(*) is very smooth & decays very quickly, so for the local/global principle also $\hat{f}(t)$ is very smooth & decays very quickly.

<u>Proof:</u> $f'(x) = -2ax \cdot e^{-ax^2}$

$f(x)$ is a solution of the ODE $2axf + f' = 0$ $\Big\{$ this is ALSO Separable Variable OD

(linear ODE, 1st order, non-constant coeff's)

Let's take the F.T. of this ODE $2a\cancel{i}(\hat{f}') + \cancel{i}t\hat{f} = 0$

$\hat{f}(t) = g(t)$ Let's solve $2ag' + tg = 0$

Separation of variables $2a\dfrac{dg}{dt} = -tg$ $\quad \dfrac{dg}{g} = -\dfrac{t}{2a}dt$

$$\int \frac{dg}{g} = -\int t\,dt \cdot \frac{1}{2a}$$

$\log|g| = -\dfrac{1}{2a}\cdot\dfrac{t^2}{2} + c \qquad |g(t)| = e^{-t^2/4a}\cdot \underbrace{e^c}_{>0}$

$g(t) = \alpha \cdot e^{-t^2/4a}$

$\equiv \hat{f}(t)$ (QDE)

<u>Lemma</u> $\boxed{\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}}$ DE MOIVRE

unfortunately $F(x)$ such that $F'(x) = e^{-x^2}$ <u>is not</u> elementary ($Erf(x)$)

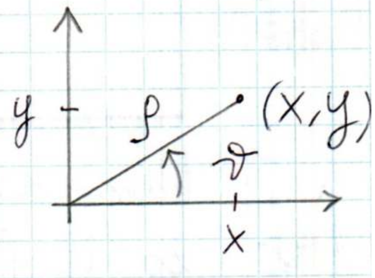
<u>Trick</u> $I = \int_{-\infty}^{+\infty} e^{-x^2}dx \qquad I^2 = \int_{-\infty}^{+\infty} e^{-x^2}dx \int_{-\infty}^{+\infty} e^{-y^2}dy =$

$$= \iint_{\mathbb{R}^2} e^{-x^2-y^2}dx\,dy =$$

<u>Polar coordinates</u> $x = \rho\cos\vartheta$

$y = \rho\sin\vartheta$

$x^2 + y^2 = \rho^2$



$$= \iint_{\mathbb{R}^2} e^{-\rho^2}\underbrace{\rho\,d\rho\,d\vartheta}_{\text{infinitesimal area element in polar coordinates}} = \int_0^{2\pi}d\vartheta\int_0^{+\infty}\underbrace{e^{-\rho^2}\rho\,d\rho}_{(-\frac{1}{2})e^{-\rho^2}d(-\rho^2)}$$

$$= 2\pi \left[-\frac{1}{2} e^{-\rho^2} \right]_0^{+\infty} = \frac{2\pi}{2} = \pi = I^2$$

$$\Rightarrow I = \sqrt{\pi} \quad (QED)$$

We know that $\hat{f}(t) = \alpha \cdot e^{-t^2/4a}$

in particular $\hat{f}(0) = \alpha e^0 \Rightarrow \alpha = \boxed{\hat{f}(0) = \int_{-\infty}^{+\infty} f(x) \underbrace{e^{-i0x}}_{\equiv 1} dx}$

(the integral of f on $\mathbb{R}$ coincides with the number $\hat{f}(0)$)

Therefore $\alpha = \int_{-\infty}^{+\infty} e^{-ax^2} dx \underset{\left(\text{change } x = \frac{x'}{\sqrt{a}}\right)}{=} \int_{-\infty}^{+\infty} e^{-a\frac{x^2}{(\sqrt{a})^2}} \frac{dx}{\sqrt{a}} =$

$$= \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{\sqrt{a}} = \sqrt{\pi/a} \quad (QED)$$

> If f and f' are in $L^1(\mathbb{R})$, then $\hat{f}$ in particular is continuous
> Also $\hat{f}(t) = o(1/t)$ as $t \to \pm\infty$

N.B. By Riemann-Lebesgue theorem (already stated, not proven, see page 24), we know that if $f \in L^1(\mathbb{R})$ then $\hat{f}$ is continuous on $\mathbb{R}$ and $\to 0$ as $t \to \pm\infty$; with the extra assumption $f' \in L^1$, we get that $\hat{f}(t) \to 0$ "faster" than $1/t$

We're only applying the Riemann Lebesgue theorem on the derivatives of f

> Def. $g(t) = o(h(t))$ as $t \to \pm\infty$
> means $\lim_{t \to \pm\infty} \frac{g(t)}{h(t)} = 0$

this property comes from $[f'(x)]^\wedge(t) = it\hat{f}(t)$

$it\hat{f}(t) = \frac{\hat{f}(t)}{1/t} i$

so if $f' \in L^1(\mathbb{R})$, then $\underset{(\infty \cdot 0)}{it\hat{f}(t)} \in C_0(\mathbb{R})$ (is continuous) and $\to 0$ as $t \to \pm\infty$.

More generally (iterating...) assume that $f, f', f'', \dots, f^{(m+1)} \in L^1(\mathbb{R})$, then $f \in C^m(\mathbb{R})$ { f has a smoothness of m continuous and $\begin{pmatrix}\text{using}\\ \text{F.T.}\end{pmatrix} \Rightarrow \hat{f}(t) = o\left(\frac{1}{t^{\underbrace{m+1}}}\right)$ as $t \to \pm\infty$

DECAY ORDER!

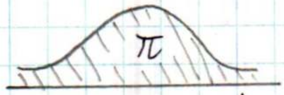
So, smoothness of f is mapped by F.T. into decay at ∞ of $\hat{f}$.

Corollary: If $f \in C^{\infty}(\mathbb{R})$ (with $f, f', f'', \ldots \in L^1(\mathbb{R})$) then $\hat{f}(t) \to 0$ faster than any $1/t^N$

For example, the Gaussian function $f(x) = e^{-ax^2}$ is in $C^{\infty}(\mathbb{R})$ and all of its derivatives are of the form $e^{-ax^2} \cdot$ polynomials $\to 0$ quickly $f, f', f'', \ldots \in L^1(\mathbb{R})$

$\hat{f}(t) = \sqrt{\pi/a}\, e^{-t^2/4a}$ decays faster than $1/t^N$ for $\forall N$

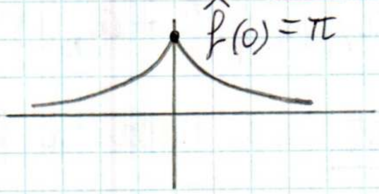
For example $f(x) = \frac{1}{1+x^2}$



$\boxed{\hat{f}(0) = \int_{-\infty}^{+\infty} f(x)\, dx}$

REMEMBER

$f \in C^{\infty}$ $\hat{f}(t) = \pi e^{-|t|}$



$e^{-|t|} \to 0$ as $t \to \pm\infty$ faster than any $1/t^N$ (which reflects the C^{∞} smoothness of f) BUT its order 2 decay is reflected in a discontinuity of $(\hat{f})'$!

Vice-versa, if f and $xf \in L^1(\mathbb{R})$, then $\hat{f}(t) \in C^1(\mathbb{R})$ and $\frac{d}{dt}\hat{f}(t) \to 0$ as $t \to \pm\infty$

More generally, if $f, xf, x^2 f, \ldots, x^m f \in L^1(\mathbb{R})$ $\Rightarrow \hat{f} \in C^m(\mathbb{R})$ and $\frac{d^k}{dt^k}\hat{f}(t) \to 0$ as $t \to \pm\infty$ for $k = 0, 1, 2, \ldots, m$

DECAY AT ∞ OF $f(x)$ IS MAPPED INTO SMOOTHNESS OF $\hat{f}(t)$.

Local properties of f are mapped into global properties and vice-versa.

Convolutions

$$\boxed{(f * g)_{(x)} \overset{\text{def.}}{=} \int_{-\infty}^{+\infty} f(x-y)\, g(y)\, dy}$$

N.B. For each fixed translation $x \in \mathbb{R}$ it is an integral of a product. Once the integral is done, x becomes the new variable.

Both in ODE and PDE theory, convolutions come up naturally. For example, given a linear ODE of order n:

$$a_n(x) y^{n} + a_{n-1}(x) y^{(n-1)} + \ldots + a_1(x) y' + a_0(x) y = f(x)$$

first we need to find n linearly independent functions $y_1(x), y_2(x), \ldots, y_n(x)$ such that the linear combination:

$y_{hom}(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$ is the general solution of $a_n(x) y^{(n)} + \ldots + a_1(x) y' + a_0(x) y = 0$ (associated homogeneous linear ODE)

The full solution has the structure

$$y_{TOT} = y_{hom} + \eta(x)$$

It can be proven that the 2nd term

$$\boxed{\eta(x) = \int_{-\infty}^{+\infty} \mathcal{K}(x-y)\, f(y)\, dy}$$

is given by a convolution where the "kernel function" $\mathcal{K}(x)$ is given by a formula that contains $y_1(x), y_2(x), \ldots, y_n(x)$ (Wronskian Matrices).

A PDE example (we'll go back to it later...) (page 51R)

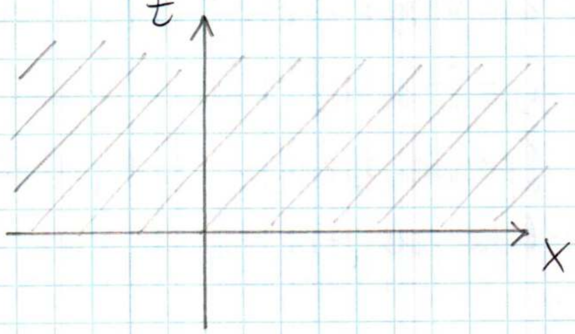
HEAT EQUATION $\boxed{\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}}$

$x \in \mathbb{R}$
$t > 0$

unknown function
$u \equiv u(x,t)$

with $\boxed{u(x,0) = u_0(x)}$

GIVEN function of 1 variable



Notation $u_x = \dfrac{\partial u}{\partial x}$

$u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$

$u_{xyy} = \dfrac{\partial}{\partial x} \dfrac{\partial^2}{\partial y^2} u$

We'll see that the solution to this problem is a convolution of a suitable gaussian with $u_0(x)$. To prove it we will use the Fourier Transform.